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MATHEMATICS

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MATHEMATICAL REVIEWS

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References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol. 22, No. 12A December, 1961 Reviews 12017–12545

GENERAL

See also 12030, 12506, B13093.

12017:

Heisenberg, Werner; Schrödinger, Erwin; Born, Max; Auger, Pierre. ★Discussione sulla fisica moderna. Quattro conferenze organizzate dalle "Rencontres internationales de Genève", 1952–1958; traduzione dal tedesco e dal francese di Adolfo Verson. Biblioteca di cultura scientifica, 59. Paolo Boringhieri, Torino, 1960. v + 131 pp. (1 plate) L. 1200.

The articles of Heisenberg and Schrödinger are reviewed below. The articles of Born ["Riflessioni d'un uomo di scienza europeo", pp. 77–100] and Auger ["I metodi e i limiti della conoscenza scientifica", pp. 101–131] are of a general nature.

12018:

Heisenberg, Werner. La scoperta di Planck e i problemi filosofici della fisica atomica. Discussione sulla fisica moderna, pp. 3–33. Boringhieri, Torino, 1960.

The discoveries of modern science, especially those of atomic physics, raise many problems which demand a philosophical interpretation and which, in turn, affect our outlook on the structure of matter. In particular, they seem to militate against the materialism of XIXth century physics. In his lecture Heisenberg discusses some of the steps in the development of modern physics and describes the demands which have to be met by a fundamental theory required to determine the structure of the world. The basic discovery of Planck consists in introducing a scale of magnitudes into nature, the microcosmic laws standing in opposition to the intuitive conception of it. The unique determination of phenomena in classical physics is modified by the relation of indeterminacy, although this represents only a correction to causal theories. Above all, we are forced to abandon an objective idea of reality, not in the sense that it does not exist but, merely, that the observer cannot be left out of the account. The probability waves arising from Born's statistical interpretation of the field phenomena can be regarded as a quantitative expression of Aristotle's "dynamis" with an intermediate value of its truth content suggestive of indeterminacy. The materialistic concept of absolute units of matter is lost, the building bricks of the world, the elementary particles, appearing as an abstraction from observable entities. The high-energy experiments indicate that while the elementary particles are not immutable, neither do they seem to be composite. In fact, they are only forms under which energy, which is the substance of reality, manifests itself.

But the multiplicity of species of elementary particles and their abstract character suggest that they should possess a mathematical structure expressed in a fundamental law of nature. Of this we demand properties of symmetry contained in Lorentz invariance under the unitary group of transformations and under parity. In addition, the law must contain three constants of measure: Planck's constant, the velocity of light and a constant of length. Their numerical value, however, is arbitrary. The above conditions are well known. It is of interest, however, that Heisenberg supports the belief in the possibility of such a law, reminiscent of the Fundamental Theory of Eddington, although less platonic in its philosophical outlook. Nevertheless, he considers the elementary particles to be analogous to Plato's representation of the elements, water, fire, earth and air, by regular solids, whose idea is contained in their mathematical form alone.

Also it is of interest to the philosophical interpretation of science to mention the parallel between physics and the history of a language on the one hand, and the mathematical interpretation of it and of grammar on the other. This was pointed out during the discussion by G. Devoto. *A. H. Klotz* (Newcastle upon Tyne)

12019:

Schrödinger, Erwin. L'immagine attuale della materia. Discussione sulla fisica moderna, pp. 35–75. Boringhieri, Torino, 1960.

The wave-particle duality of material entities, presupposed by the quantum theory, implies a conflict between the scientific outlook and the commonsense view of reality. The author attempted to escape this dilemma by proposing an interpretation of wave mechanics based on the assumption that all microscopic phenomena could be explained exclusively by the wave field. In his opinion, it is necessary to distinguish the longitudinal and the transverse wave effects, the former referring to the single particle aspect. The normals to a wave surface correspond then to the trajectories of the particles whose structure is determined by a wave packet section of the dispersing wave trains. This implies the concept of indeterminacy, while the discontinuous quantum jumps are referred to resonance frequencies. The author's particles are correctly deprived of the individuality which can be given to a wave field whose form and structure is well determined, just as it is in bodies persisting in time. The conservation properties of mass and of charge are statistical concepts depending on the law of large numbers.

The discussion which followed the author's exposition centred around the problem of the meaning of Bohr's principle of complementarity and the point at which individuality could be ascribed to material bodies. There

seemed to be agreement on the fact that the latter depends on the philosophical concept of form, in spite of the aristotelian thesis that matter is the principle of individuality. L. Rosenfeld claimed that theoretical thinking in physics proceeds on dialectic lines which do not exclude the possibility of contradictory statements. It is a little difficult to see, in that case, why there should be so many attempts to free the theories from the so-called paradoxes.

The point of view of the late Professor Schrödinger is, in a sense, complementary to that of Bohm, whose work admitted the particle aspect of reality only.

A. H. Klotz (Newcastle upon Tyne)

12020:

Maros dell'Oro, Angiolo. Il significato fisico delle formule di Lorentz. Considerazioni epistemologiche. Archimede **12** (1960), 57–64.

12021:

Krull, Wolfgang. Über die Endomorphismen von total geordneten Archimedischen Abelschen Gruppen. Math. Z. **74** (1960), 81–90.

Eudoxus' algebra of proportions, as expounded in Euclid's Book V, is given a modern interpretation in terms of endomorphisms of a (simply) ordered (Archimedean) Abelian group. The discussion has historical and expository interest in connection with the basic question: "What is a quantity (Grösse)?"

G. Birkhoff (Cambridge, Mass.)

12022:

Henmueller, Frank; Menger, Karl. What is length? Philos. Sci. **28** (1961), 172–177.

Let Δ be a class of extramathematical objects. Let $\sigma \in \Delta$ and $u\sigma$ a number called the value of u for σ. The fluent u is defined as the class of all ordered pairs $(\sigma, u\sigma)$. A 2-place fluent V is defined as the class of all pairs $[(\rho, \sigma), V(\rho, \sigma)]$, where (ρ, σ) is an element (ordered pair) in a class of extramathematical objects and $V(\rho, \sigma)$ is a number paired with (ρ, σ). The authors interpret length ratio in terms of 2-place fluents and discuss some of their mathematical properties. They conclude with some philosophical remarks.

K. S. Miller (New York)

12023:

Adler, Irving. ★Thinking machines: A layman's introduction to logic, Boolean algebra, and computers. The John Day Co., New York, 1961. 189 pp. $4.00.

This is a popular account of the use of computers and their application to logical rather than numerical problems. It discusses such topics as the use of Boolean algebra in the design of circuits, and although the treatment is necessarily superficial, the book is one of the more successful attempts at the "popularisation of science".

H. Kolsky (Providence, R.I.)

12024:

★Les mathématiques de l'ingénieur. Comptes rendus du Congrès International, Mons et Bruxelles, 9–14 juin 1958. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, volume hors série. Mons, 1958. 438 pp. + Atlas of 23 pp. 300 F.

The articles in this collection are being reviewed individually.

12025:

Fazekas, Francis. Some remarks about the higher teaching of engineering mathematics in Hungary. Les mathématiques de l'ingénieur, pp. 417–425. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

12026:

Garreau, G. A. Technical education in England and Wales. Les mathématiques de l'ingénieur, pp. 426–428. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

12027:

Girault, Maurice. La formation mathématique de l'ingénieur. Quelques aspects actuels du problème. Les mathématiques de l'ingénieur, pp. 429–435. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

HISTORY AND BIOGRAPHY

12028:

Belaval, Yvon. ★Leibniz critique de Descartes. Bibliothèque des Idées. Librairie Gallimard, Paris, 1960. 559 pp. 22.50 NF.

Einleitend schildert der Verfasser, wie Leibniz mit Descartes' Schriften bekannt wurde. Im I. Teil behandelt er die beiderseitigen Methoden, im II. Teil das von jedem zu Grunde gelegte mathematische Modell und im III. Teil die Anwendung auf die damals als entscheidend angesehenen physikalischen Hauptfragen. Die sehr sorgfältige Bibliographie führt gut; leider kann das ziemlich ausführliche Inhaltsverzeichnis das fehlende Personen- und Sachregister nicht ersetzen.

J. E. Hofmann (Guenzburg)

12029:

★Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers. Teil I: Der Briefwechsel L. Eulers mit G. F. Müller, 1735–1767. Herausgegeben und eingeleitet von A. P. Juškevič und E. Winter unter Mitwirkung von P. Hoffmann. Deutsche Akad. Wiss. Berlin. Quellen und Studien zur Geschichte Osteuropas, Bd. III, Teil I. Akademie-Verlag, Berlin, 1959. ix + 327 pp. (2 plates) DM 32.50.

The 250th anniversary of L. Euler's birthday in 1957 gave rise to a number of studies—not only on his life and his works but also on the relations between the academies in St. Petersburg and Berlin where he spent 57 years of his life. But while the immense task of reprinting Euler's published works, which was begun 50 years ago, has now filled about 75 large volumes and is finally nearing completion, there is still a wealth of material such as his letters waiting for publication.

The present volume presents for the first time the extensive correspondence (in German) between Euler and the historian and geographer Gerhard F. Müller who was secretary of the Academy in St. Petersburg from 1754 to 1765, i.e., some of the years during which Euler was living in Berlin as president of the Section for Mathematics and Physics of the Prussian Academy (1741–1766). The bulk

of the more than 200 letters was written during those 11 years, but 12 originated in 1734–1742, and 7 fall into the years 1766–1767. Though the correspondence is not at all mathematical, it will be indispensable for any future biographer of Euler. His deep concern for, and his valuable assistance to, the Russian Academy even in the years of his absence from St. Petersburg reveal themselves in these letters: when he proposes and negotiates with prospective members; when he searches for books and maps all over Europe; when he suggests problems for the scientific prize competitions; when he reviews, criticizes or prepares for edition articles submitted to the St. Petersburg Academy; when he accommodates outstanding Russian students in his home and prepares them for a career in Petersburg; when he works out plans for a reorganization of that academy, to mention only his outstanding activities aside from his scientific research which filled the journals of both academies for many years. Towards the end of the correspondence Euler's plans to return to St. Petersburg take a good deal of space, throwing light on the unfavorable conditions in Berlin.

A summarizing introduction, plenty of annotations and references, a bibliography of mostly Russian titles, and a carefully prepared index of names give any assistance the reader may ask for. *C. J. Scriba* (Toronto)

12030:

Bierens de Haan, D. ★Bibliographie néerlandaise historique-scientifique des ouvrages importants dont les auteurs sont nés aux 16e, 17e et 18e siècles, sur les sciences mathématiques et physiques, avec leurs applications. B. de Graaf, Nieuwkoop, 1960. 424 pp. 50 guilders; $13.50.

This important bibliography of more than 5700 entries was first published in 1881–1882 as a series of articles in the 'Bolletino di bibliografia e di storia delle scienze mathematiche e fisiche', and appeared separately in Rome in 1883. An unchanged reprint of the Rome edition, which has long been out of print, is now offered.

The listing has been arranged alphabetically according to authors; where available, some biographical information has been added. Very valuable is the index in which the total area of science has been divided into 115 fields. Under the 38 headings belonging to mathematics one finds the wealth of more than 1000 names, many occurring several times. A special table of 30 pages lists the printers and publishers of the works included in the bibliography: about 1400 names altogether.

The Rome edition was full of misprints; 20 columns of corrections were attached in an appendix. There ought to be a method to avoid the reproduction of 1000 mistakes in a single volume. It would at least have been desirable to give these corrections as footnotes on the same page; as it is now, the casual user is bound to overlook them.

C. J. Scriba (Toronto)

12031:

Molnár, J. Bemerkungen zu den Grundkonstruktionen der hyperbolischen Ebene. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 31–32.

V. F. Kagan, in a Russian edition of J. Bolyai's *Appendix* [Moscow, 1950; p. 118] takes issue with Bolyai's method of constructing a parallel. The author meets Kagan's objection by showing the inherent simplicity of Bolyai's proof. *S. R. Struik* (Cambridge, Mass.)

12032a:

Licis, N. The problem of the scientific and philosophical significance of the geometry of Lobačevskiĭ. Latvijas PSR Zinātņu Akad. Vēstis **1960**, no. 3 (152), 33–44. (Russian)

12032b:

Licis, N. Natural-scientific and philosophic ideas of N. I. Lobačevskiĭ in the development of his notions about space and time. Latvijas PSR Zinātņu Akad. Vēstis **1960**, no. 8 (157), 5–18. (Russian. English summary)

It is convenient to review these two papers simultaneously since whole paragraphs are verbally identical, only the order of the remarks being changed and a different emphasis put. Lobačevskiĭ is said to have fought a heroic battle against dogmatism and conservatism and to have erected a barrier between the typical bourgeois philosophy of Kant—in fact a repetition of Plato's concepts—and Mach and the metaphysical narrowness (of mind) demonstrated by Newton in his views on space and time. It is stated in the first paper that only because of Lobačevskiĭ's non-euclidean geometry could the theory of relativity be developed, whereas the second paper asserts that the priority of the building of the mathematical foundations of the theory of relativity rightly belongs to him. In the second paper attention is drawn to his remarks on particles and waves in the theory of light, with which the author also seems to indicate a claim of priority for Lobačevskiĭ in modern quantum mechanics. Though the author points to the necessity of continuous revision of scientific concepts, the present reviewer obtained the impression that the author forgot to require the same for using a terminology and exact quotations. So, e.g., the fifth postulate of Euclid is "quoted" in a way which is with no rhyme or rhythm to Euclid's formulation. Lobačevskiĭ's hyperbolic geometry and its relation to other geometries are described in a way which suggests that the imbedding of spaces is unknown to the author. The alleged materialistic views of Lobačevskiĭ are indicated to be in accordance with statements of Lenin, Lomonosov and Engels. Hitherto the reviewer was of the opinion that Lobačevskiĭ's remarks on particles and the wave aspects of light were related to Newton's corpuscular theory and Huygens' wave theory. It seems highly improbable that relations to modern quantum mechanics exist. The reviewer thinks that Kant's antinomies showed to the "western philosophers" clearly that one must carefully realise that the same word should not be used with slightly different meanings. So the reviewer appreciated these papers in their playing with different meanings of "matter", "material", "materialistic", "relative", "relativity", "geometry", "physics" and the attacks on conservatism and dogmatism, until in the second paper (p. 14) he read: "Geometry studies the geometrical properties of space, determined by distribution and motion of masses. Therefore every tendency to connect physics to geometry by using non-riemannian geometries leads to identification of matter and extension, which contradicts Lenin's doctrine on matter and the forms of its motion." For even if one

admits the shifts in significance of the terms used, such a statement must be seen as purely dogmatic.

E. M. Bruins (Amsterdam)

12033:

LaSalle, J. P. A report on Soviet mathematics. Amer. Math. Soc. Not. **8** (1961), 25–29.

Announcement of the completion of a survey, together with certain conclusions of the survey panel, of recent Soviet contributions to the following areas of mathematics: algebra, control and stability theory, functional analysis, numerical analysis, partial differential equations, probability and statistics, topology.

12034:

Tenca, Luigi. I presunti contrasti fra Evangelista Torricelli e Vincenzio Viviani. Period. Mat. (4) **38** (1960), 87–94.

12035:

Huber, Friedrich. ★Daniel Bernoulli (1700–1782) als Physiologe und Statistiker. Basler Veröffentlichungen zur Geschichte der Medizin und der Biologie, Fasc. VIII. Benno Schwabe & Co., Basel–Stuttgart, 1959. 103 pp. (1 plate) $4.00.

Though justly most famous for his achievements in mathematical physics, Daniel Bernoulli was training in medicine, occupied for a time the chair of anatomy and botany at Basel, and made significant contributions to physiology and medical statistics. This book describes these lesser known aspects of his life and relates them to his other work. *Kenneth May* (Northfield, Minn.)

12036:

Cardano, Gerolamo. ★The book on games of chance (Liber de ludo aleae). Translated by Sydney Henry Gould; foreword by Samuel S. Wilks. Reprinted from "Cardano: The gambling scholar", by Oystein Ore (Princeton University Press, 1953). Holt, Rinehart and Winston, New York, 1961. v+57 pp.

A reprinting of "The book on games of chance", which appeared in O. Ore's *Cardano: The gambling scholar* [Princeton Univ. Press, Princeton, N.J., 1953; MR **14**, 609]. Many footnotes in the present volume must be sought in Ore's original text.

12037:

Oresme, Nicole. ★Quaestiones super geometriam Euclidis. Edited by H. L. L. Busard. Janus, suppléments, Vol. III. E. J. Brill, Leiden, 1961. xiv+179 pp. (2 plates) 17.50 guilders.

Nicole Oresme (?1323–1382) was a French priest, philosopher and mathematician who played a not inconsiderable role in the history of science. As far as we know, he was the first author to introduce graphic representations of variable quantities. Among his works the *Quaestiones on Euclid's Elements* are mentioned; this treatise was never published. Dr. Busard has now filled up this lacuna by a modern critical edition. He gives us a correct Latin text and provides this with a commentary in the form of a paraphrase which comes near to a translation. He has added figures to facilitate the understanding of Oresme's reasonings. Ten of the *Quaestiones* deal with topics current in fourteenth-century scholastic mathematics, eight of them contain Oresme's theory about the graphic representation of qualities and velocities, the remaining three treat of the concept "angle".

E. J. Dijksterhuis (Bilthoven)

12038:

Ляпунов, А. М. [Lyapunov, A. M.] ★Собрание сочинений, Том 3 [Collected works, Vol. 3]. Izdat. Akad. Nauk SSSR, Moscow, 1959. 375 pp. 16.90 r.

[For volumes 1 (1954) and 4 (1959), see MR **19**, 109; **21** #5542.] This volume contains six memoirs, dealing with equilibrium figures of rotating fluids and the form of planets. Five of these were originally published between 1884 and 1908, the sixth appeared in 1930.

H. A. Antosiewicz (Los Angeles, Calif.)

LOGIC AND FOUNDATIONS

See also 12062, B12748, B12921, B12922.

12039:

Stenius, Erik. ★Wittgenstein's Tractatus: A critical exposition of its main lines of thought. Cornell University Press, Ithaca, N.Y., 1960. xi+241 pp. $5.00.

The *Tractatus* has been one of the most influential books in the history of modern philosophical and logical thought. Since it is revolutionary in aim, it has taken time for the necessary distance to be built up which would make an objective commentary possible. The present volume is one of the first to do this in anything like an adequate way. Wittgenstein was not treated well by his translator, and it is therefore of no little importance that the author throughout his book gives new translations and paraphrases of disputable terms (such as "Sachverhalt" and "Tatsache"); and indeed each of the chapters contains (like "nodal points") statements by the commentator of Wittgenstein's thesis as it appears at the point under discussion.

Many of the most difficult concepts of the *Tractatus* are considered with great care and by means of illuminating models. Thus, the idea of "logical space" is investigated through the use of a spatial model; "Hume's thesis" of the atomicity of all states of affairs in this way receives an illuminating interpretation, and the author often manages to insert some important corrections and novel approaches to Wittgenstein's point of view. The notion of the "logical form" of a "thing" is interpreted as its category (p. 71), but one of the more interesting (if debatable) discoveries is that the mysterious "things" are not only individual objects but also predicates (p. 63). This view makes possible an account of the function of a sentence as a "picture". The chapters which deal with Wittgenstein's concept of picture, and of a sentence as a picture, are indeed the central ones in this volume, and will go far to show how much important sense may still be extracted from a "picture-theory of meaning". One of the main points here is that in a sentence-token such as "*mSe*" ("the moon is smaller than the earth") whilst "*m*" and

"e" may be interpreted as names, S must be interpreted as an "S-relation" (pp. 131–132), in order to get a genuine isomorphism between a sentence-token and the fact described by it. The philosopher of science will be interested also in the many illuminating applications of the concept of "world-picture" to scientific constructs (p. 27), or again, the relation between the framework of a world description and experience, as illuminated by the method of relativity theory (p. 86). And the general philosopher will find himself confronted once again by the deeper problem of how a sentence, as such composed of different simple signs, comes to be understood as a connection of such signs (p. 127). Wittgenstein's notion of a method of projection has not always found a receptive echo from those who have harped on more superficial ideas of structural similarity (p. 128). The development of some of these themes (such as the one just mentioned, in the *Philosophical investigations*, I.22) is also made apparent. New students of Carnap will also be interested in Stenius's criticism of the function of the theory of the "material" and the "formal mode of speech", and the failure of the latter as a translation from the former (pp. 182–186).

The last chapter illustrates the similarities between the doctrines of Wittgenstein and of Kant. This will have the added advantage of helping those continental philosophers to whom 20th century English philosophy is still a closed book, to approach Wittgenstein's ideas in the *Tractatus* from the side of a more familiar terminology.

G. Buchdahl (Cambridge, England)

12040:

Guillaume, Marcel. Sur une propriété remarquable du système de Bourbaki. C. R. Acad. Sci. Paris **250** (1960), 1776–1777.

The property in question is that if M is any model of Bourbaki's system of set-theory then all theorems true in the model remain true when the domain is restricted to the nameable elements of M. Definitions of the relevant concepts (model, nameable, etc.) and semantic proofs leading to the result are indicated. It should be mentioned that this property holds for most systems of set-theory.

L. N. Gál (Minneapolis, Minn.)

12041:

Porte, Jean. Un système pour le calcul des propositions classiques où la règle de détachement n'est pas valable. C. R. Acad. Sci. Paris **251** (1960), 188–189.

This paper gives a simple example of a system P' for which we have the rule: if $\vdash x$ and $\vdash(x \supset y)$, then $\vdash y$; but in which, in general, we do not have the rule: $x, x \supset y \vdash y$. The system in question is a reformulation of Church's system P [*Introduction to mathematical logic, Vol. I*, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 631; § 57]. In this reformulation the axioms are the double negations of the axioms of P and the rules of inference are: (R1) $\neg\neg x, \neg\neg(x \supset y) \vdash \neg\neg y$, and (R2) $\neg\neg x \vdash x$.

L. N. Gál (Minneapolis, Minn.)

12042:

Smullyan, Raymond M. ★Theory of formal systems. Annals of Mathematics Studies, No. 47. Princeton University Press, Princeton, N.J., 1961. xi+142 pp. $3.00.

This monograph is the most elegant exposition of the theory of recursively enumerable (r.e.) sets in existence. The treatment is based on elementary inductive definitions of r.e. sets (Post's production rules) and is formulated for arbitrary alphabets and not only for arithmetical ones. This avoids tiresome numberings. All the well-known results on r.e. sets are given, including variants and refinements; in particular, generalizations of Gödel's first incompleteness theorem, Rosser's variant, enumeration and normal form theorems, recursion theorems, results on creative sets, etc. Here the improvement over previous expositions is achieved by a systematic analysis of what predicates have to be 'representable' (and in what sense) for the various known theorems to hold. Another striking improvement is due to the use of so-called constructive arithmetic and rudimentary predicates instead of primitive recursive ones for the existential definition of r.e. predicates. The constructive-arithmetic ones are obtained by bounded quantification from addition and multiplication and, since they do not contain exponentiation, are even more restrictive than Kalmar's elementary predicates. The rudimentary ones are inductively defined from the concatenation relation, and therefore particularly adapted for expressing in a uniform and canonical manner the proof predicate of formal systems determined by (a finite set of) production rules, i.e., of formal systems in the original sense of the term. It may be remarked that this latter improvement has not yet been fully exploited in the monograph: in particular, it is bound to simplify immensely the proof of Gödel's second undecidability theorem because the derivability conditions of Hilbert and Bernays [*Grundlagen der Mathematik, Vol. II*, J. W. Edwards, Ann Arbor, Mich., 1944; MR **6**, 29], are much easier to verify for constructive arithmetic formulations of the proof predicate than for primitive recursive ones. Further, apart from this technical advantage, there is now the possibility of defining once and for all the notion of formal expressibility of the proof predicate for arbitrary formal systems determined by production rules: a formula $\pi(x, y)$ is said to express the (proof) predicate P in a system F if π not only strongly represents P (in the sense of p. 46), but if the implications corresponding to the production rules are formally derivable in F when π is substituted for the corresponding letters in the production rules. Of course, here a formal system is determined by the set of its production rules and not merely by its set of theorems.

In the supplement the results on r.e. sets are applied to theories in standard formalization (r.e. axiom systems in first-order predicate logic). While it is of course interesting to have incompleteness theorems derived directly from the study of r.e. sets, it seems to the reviewer that the original, more proof-theoretic, approach of Gödel and Rosser is not superseded. Thus, e.g., Rosser's variant may be regarded as an application of Gödel's original construction of a formula $(x)\neg\pi(x, q)$, where the value of the term q is the number of the formula $(x)\neg\pi(x, q)$, except that Rosser uses another definition of the proof relation, namely, $\pi(x, y) \wedge (z)_{<x} \neg\pi[z, e(y)]$ instead of $\pi(x, y)$, $e(y)$ representing the negation of y. Or, while the need for ω-consistency in Gödel's own work is discussed on p. 133 in terms of the difference between enumerability and representability, one can alternatively observe this: using the standard arithmetization of consistency (of a system satisfying the conditions for Gödel's second

theorem, e.g., Z of Hilbert-Bernays), let Z_1 denote the system $Z \cup \{\neg \mathrm{Con}\ Z\}$; then the sentence constructed by Gödel for Z_1, is equivalent (even in primitive recursive arithmetic) to Con Z_1. But this sentence is not undecidable in Z_1, since evidently $\vdash_{Z_1} \neg \mathrm{Con}\ Z_1$; Z_1 however is consistent though, of course, not ω-consistent. Also another significant difference between Gödel's Con F and Rosser's sentence R_F for a system F can be expressed proof-theoretically: while R_F follows from Con F even in primitive recursive arithmetic, $\vdash_F (R_F \rightarrow \mathrm{Con}\ F)$ does not hold. In short, the original approach showed the non-derivability of certain arithmetic formulae which have a simple syntactic interpretation (such as consistency of the system considered), which of course cannot be expected from an approach based only on the theory of r.e. sets. This fact is important for foundations, e.g., the incompleteness of formal systems with respect to intuitionistic interpretations does not follow from mere non-saturation, since there is no reason to suppose that every statement is either valid or absurd, but it does follow from the non-derivability of formulae with a suitable interpretation. A really elegant companion volume by the author on the arithmetization of syntax of formal systems (which should be feasible by the use of the author's constructive-arithmetic proof predicates) with an up-to-date exposition of Gödel's second incompleteness theorem and related results would be most welcome.

{Reviewer's note: As a treatment of r.e. sets and r.e. formal systems, the present monograph seems perfectly planned. But it appears definitely misleading to regard the r.e. character of 'known' formal systems as a kind of empirical accident (footnote 5 on p. 9): whatever defects Turing's argument may have, it is certainly a serious attempt to analyse the meaning of 'mechanical' or 'formal' method, and should not have been completely ignored. On the other hand, the author seems right in emphasizing that the rules of proof need only be r.e. and not recursive. For, in conducting a formal proof, one need only verify that given sequences of formulae constitute a proof, and never that they do not; so, r.e. rules are sufficient.}

G. Kreisel (Paris)

12043:

Успенский, В. А. [Uspenskiĭ, V. A.]. ★Лекции о вычислимых функциях [Lectures on computable functions]. Matematičeskaya Logika i Osnovaniya Matematiki. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 492 pp. 13.60 r.

This is a pleasantly discursive expository volume on elementary recursion theory, written from the 'classical' point of view, i.e., treating recursive functions simply as mathematical objects to be investigated by whatever means are at hand, constructive or otherwise, rather than as tools ancillary to the Hilbert program in the style of Kleene. It presupposes virtually nothing; the small amount of set-theory, logic and real variable required is developed in the book itself. For this reason it is well-adapted for use as a textbook in upper-division undergraduate or first-year graduate courses (in philosophy as well as in mathematics). Two semesters should be ample. It compares favorably from the point of view of lucidity of exposition with any of the three existing textbooks of recursion theory: Kleene's *Introduction to metamathematics* [Van Nostrand, New York, 1952; MR **14**, 525], Davis's *Computability and unsolvability* [McGraw-Hill, New York, 1958] and Péter's *Rekursive Funktionen* [Akademischer Verlag, Budapest, 1951; MR **13**, 421]; it contains much material not covered in any of these three books, and the ratio of words to symbols is refreshingly high. On the other hand, this reviewer feels that the choice of subjects is somewhat idiosyncratic (for example, there is no mention of functionals, hierarchy theory, degrees or unsolvable decision-problems). The treatment of those subjects which it does cover is so good, however, that it would be a pity not to translate it into English.

After an introductory motivational chapter (Chapter 1) and a digest of the set-theory (Chapter 2) and logic (Chapter 3) required in the sequel, it begins with primitive recursive functions (Chapter 4). The usual closure properties are established in a smooth and somewhat repetitive way (e.g., many properties of (relatively) primitive recursive sets are restated and re-proved for primitive recursive predicates; this type of diffuseness, here as throughout the book, is probably an advantage in the classroom). Chapter 5 deals with recursively enumerable (r.e.) sets, again chiefly from the point of view of their closure under various operations. They are defined as the projection of (two-dimensional) primitive recursive sets. ('Two-dimensional set' means 'set of pairs'; the geometrical terminology is characteristic of the book.) This order of development may seem strange to one who is accustomed to seeing general recursion introduced between primitive recursion and recursive enumerability, but in the long run it probably saves a fair amount of tedious calculation. Chapter 6 deals with partial recursive functions (defined as those obtainable from the usual initial functions by primitive recursion, composition and μ) from the point of view of their closure properties and their connection with r.e. sets. Chapter 7 is about (general) recursive functions [sets, predicates], again from the point of view of closure properties. (Recursive functions are defined as partial recursive functions in whose definition μ is applied only to (full) functions which vanish at least once, while recursive sets are defined as sets such that both they and their complements are r.e., i.e., projections of primitive recursive sets. The equivalence of 'recursive set' and 'set whose characteristic function is recursive' is then forthcoming as a (rather easy) theorem.) The chapter ends with a section containing the usual results on enumeration of r.e. sets by primitive recursive, general recursive, partial recursive, monotone, strictly monotone and one-one functions. Chapter 8 constructs a recursive function which is universal for all primitive recursive functions; it is the only part of the book which is at all 'heavy going' in the sense that the complexity of the notation greatly exceeds the complexity of the thought. There is a temptation for this to happen throughout recursion theory, and the author is to be congratulated on only letting his symbolism run away with him once. Chapter 9 constructs a universal partial recursive function and a universal (two-dimensional) r.e. set, as well as a doubly universal pair of r.e. sets [cf. Smullyan, #12042; the author's double universality is Smullyan's total double universality, except that the author's sets are two-dimensional; cf. also Mučnik, Trudy Moskov. Mat. Obšč. **7** (1958), 407–412; MR **21** #4098]. The existence of a r.e. but non-recursive set is also proved in this chapter —it seems unfortunate that this fact, surely the cornerstone of recursion theory, should be stated as an 'example' rather than a theorem (p. 258).

The next four chapters constitute the more 'exotic' and personal part of the book. Chapter 10 returns to r.e. sets in a somewhat miscellaneous way; it discusses first uniformizability, i.e., to use a terminology somewhat different from the author's, the existence of a r.e. choice set for every non-empty r.e. class of disjoint non-empty r.e. sets (this plays the same role in the author's development as does the intensional ν-operator in Kleene's; cf. Theorem XXV of the latter's book mentioned above, p. 346). Then the author turns to the 'reduction theorem' for r.e. sets [cf. Smullyan, op. cit., p. 90] and to the construction of recursively inseparable r.e. sets ('recursively inseparable' means simply 'not separated by disjoint recursive sets' and must be distinguished from 'effectively inseparable' for which see below). Chapter 10 concludes with a brief discussion of simple and hypersimple sets. (Post's original construction [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29] is used for a simple but not hypersimple set, while the easier construction of Dekker's paper [Proc. Amer. Math. Soc. **5** (1954), 791–796; MR **16**, 209] is used to obtain a hypersimple set.) Medvedev's theorem [Dokl. Akad. Nauk SSSR **102** (1955), 211–214; MR **18**, 272] characterizing hypersimple sets in terms of recursive majorizing functions is also proved; the author states that he himself obtained the same result independently of Medvedev.

Chapter 11 (Enumerations and operators) studies the recursive enumerability of the family of all partial recursive functions and of the class of all r.e. sets from a point of view close to that of Rogers [J. Symb. Logic **23** (1958), 331–341; MR **21** #2585], though it does not actually prove Rogers' isomorphism theorem; the iteration theorem [Kleene, op. cit., Theorem XXIII, or Davis, op. cit., Chapter 9, Theorem 1.2] appears for the first time here. (It is stated for r.e. predicates as well as partial recursive functions; cf. Smullyan, op. cit., pp. 67–70.) There are stated and proved a good many variants of Rice's theorem [Trans. Amer. Math. Soc. **74** (1953), 358–366; MR **14**, 713] and related results; the author's pedagogical skill shows itself here in that though he does not prove or even state the rather difficult Kreisel-Lacombe-Shoenfield theorem [C. R. Acad. Sci. Paris **245** (1957), 399–402; MR **19**, 521], he proves a good many special cases (e.g., 'no recursive function is effectively distinguishable from all other recursive functions') and related results (e.g., 'there is no non-trivial class of recursive sets which is effectively distinguishable from its complement in the class of recursive sets'; the reviewer does not remember having seen this one before). The result is that the reader will be put with a minimum of effort into a position from which he can appreciate the motivation and the methods used in current papers on this part of recursion theory. Similar features are present in most chapters of the book, and constitute a favorable augury for its use as a text. (On the other hand this is the one chapter of the book in which the author's personal idiosyncrasies run some risk of distorting the unsophisticated reader's perspective. It seems very odd to have such a relatively detailed discussion of (everywhere defined) effective operations [Myhill and Shepherdson, Z. Math. Logik Grundlagen Math. **1** (1955), 310–317; MR **17**, 1039; these are the 'operators' mentioned in the title of the chapter] and not even a mention of partial recursive functionals.) The chapter contains also some examples of recursive enumerations which are onto (the set of all natural numbers) but not 'fully effective' in the sense of Rogers [op. cit.] ('principal' in the author's sense).

Chapter 12 deals (more neatly than most treatments) with the beginnings of recursive analysis. It gives four definitions of 'recursive real number' based respectively upon Cauchy sequences, Dedekind cuts, nested intervals and decimal (more generally, n-ary) expansions. The author proves that all four definitions are (extensionally) equivalent but only the first three are effectively equivalent (in the sense that given an 'index' of a recursive real number in the enumeration arising from one definition, one can effectively calculate its 'index' in the enumeration arising from another). He also gives a necessary and sufficient condition, which this reviewer does not recall having seen elsewhere, that the definitions based on m-ary and n-ary expansions be effectively equivalent, namely, that the numbers m and n possess the same prime factors. The chapter concludes with a proof of the non-effective enumerability of the set of all recursive real numbers.

Chapter 13 deals with the 'recursivization' of some notions of set-theory. It starts with 'recursively infinite' sets (several definitions are proposed and each is shown to be equivalent, either to infinite-and-not-immune, or to infinite-and-not-hyperimmune). It then discusses the first properties of productive sets and effectively inseparable sets. (There is some slight history in connection with this last topic, which the reader should be made aware of in view of conflicting terminological usage. Mučnik, in the paper referred to above, raises the question of a 'natural' definition of effective inseparability for non-disjoint pairs of sets. Smullyan [op. cit., p. 119] attempts such a definition but it has the odd consequence that it makes effectively inseparable any two sets whose intersection is not r.e. The author's (p. 395) seems much better, namely: The sets M_1, M_2 are effectively inseparable if given (indices of) any r.e. sets K_1, K_2, we can effectively compute a number which belongs to one of the four sets $K_1 \cap K_2$, $\overline{K_1 \cup K_2}$, $M_1 \setminus K_1$, $M_2 \setminus K_2$, i.e., a 'witness' to the fact that $[K_1, K_2]$ either does not partition the integers at all, or partitions them wrongly. Both Smullyan's definition and the author's reduce to the classical one for disjoint M_1, M_2 and (presumably) the author's definition reduces to Smullyan's for $M_1 \cap M_2$ r.e.

The last chapter (14, on machines) distinguishes machines of Type I and of Type II (roughly speaking, finite automata without input and Turing machines respectively). The author proves that a function is computable by a Type II machine if and only if it is partial recursive, and that Type II machines are behaviorally equivalent to classical Turing machines. The reviewer has the impression that his proof requires somewhat less bookkeeping than earlier ones.

If note is taken of the somewhat adverse criticisms made in regard to Chapters 8 and 11, it would be possible to give a smoother and more suggestive course on the elements of recursion theory from this book than from any of its competitors. With this in mind, its speedy translation into English is recommended.

{Erratum: p. 395, the displayed formula near the bottom of the page: the second $\cap$ should be $\cup$.}

J. Myhill (Stanford, Calif.)

12044:

Nelson, D. Negation and separation of concepts in constructive systems. Constructivity in mathematics:

Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 208–225. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959.

A regular transformation * on formulas of a formal system L_1 to its subsystem L_2 is such that (1) for every A of L_1, $\vdash_1 A \equiv {}_1A^*$, (2) for every A, B of L_1, if $\vdash_2 A^* \equiv {}_2B^*$, then $\vdash_1 A \equiv {}_1B$, (3) for E a prime formula, E^* is E. If there is no regular transformation from L_1 to L_2, then $\equiv_1$ in L_1 allows distinctions which are not regularly represented by $\equiv_2$ in L_2; briefly L_1 allows more distinctions than L_2. Three systems S, N, I are compared. S is the system indicated by the author [Bull. Amer. Math. Soc. **60** (1954), 72]; it is obtained from Schütte's system for arithmetic [Math. Ann. **122** (1950), 47–65; MR **12**, 233] by the addition of axioms for strong negation $\neg$ and by weakening $A \supset B$ in some places to $A \supset (A \supset B)$. N results from S by restoring $A \supset B$ in these places and by deletion of the contraposition axiom. The intuitionistic calculus I results from N by deletion of all formulas containing $\neg$. It is shown that S allows more distinctions than N; in particular, $F(a)$ & $\neg F(a)$, where $F(a)$ is not recursive, is not equivalent in S to $a+1=a$. In the second part of the paper a realizability definition R is given, for which A & $\neg A \supset B$ is not valid, and a system M such that any formula deducible in M from R-realizable formulas, is R-realizable. Intuitionistic arithmetic is interpretable in M. If $1=0$ is added as an axiom to M, not every formula is provable in the new system.

A. Heyting (Amsterdam)

12045:

Kogalovskiĭ, S. R. A general method of obtaining the structural characteristics of axiomatized classes. Dokl. Akad. Nauk SSSR **136** (1961), 1291–1294 (Russian); translated as Soviet Math. Dokl. **2**, 196–199.

Various structural characterizations of classes of models in UEC_Δ, $UEUC_\Delta$, $EUEC_\Delta$, etc., are proved. Most of the results are formulated in terms of Chang's notion of n-covering [Proc. Amer. Math. Soc. **10** (1959), 120–127; MR **21** #2576].

E. Mendelson (New York)

SET THEORY

See 12294.

COMBINATORIAL ANALYSIS

12046:

Zech, Theodor. Wiederholungsfreie Folgen. Z. Angew. Math. Mech. **38** (1958), 206–209.

The author constructs an example of an infinite word $f_1 f_2 f_3 \cdots$, where each f_i ($i = 1, 2, 3, \cdots$) is one of the three symbols a, b, c, which has the property that for no finite word w does ww occur in the constructed word. The author uses expansions of integers to the base 12 to produce his example. (An earlier construction was given by Morse and Hedlund [Amer. J. Math. **60** (1938), 815–866], using two symbols, such that no word www occurs.) The author remarks that if the rules of chess were modified, the existence of these infinite sequences would imply the existence of arbitrarily long chess games without draws.

E. C. Posner (Claremont, Calif.)

12047:

Schensted, C. Longest increasing and decreasing subsequences. Canad. J. Math. **13** (1961), 179–191.

In part I it is proved that there is a one-to-one correspondence between finite sequences of distinct integers $x_1, \cdots, x_n$ and ordered pairs of standard Young tableaux of the same shape—the first, called the P-symbol, containing the integers $x_1, \cdots, x_n$ and the second, the Q-symbol, containing the integers $1, 2, \cdots, n$. Further, the number of columns [rows] in the P-symbol is equal to the length of the longest increasing [decreasing] subsequence. Since the number of standard tableaux of a given shape is known, the number of sequences of distinct integers having a longest increasing subsequence of length α and a longest decreasing subsequence of length β can be computed. Part II extends these results to sequences with repetitions.

{Without the insertion of the word "longest" before "increasing" and "non-decreasing" the second correspondence mentioned on p. 189 is not "one-to-one", as stated.}

D. E. Rutherford (St. Andrews)

12048:

Walker, R. J. An enumerative technique for a class of combinatorial problems. Proc. Sympos. Appl. Math., Vol. 10, pp. 91–94. American Mathematical Society, Providence, R.I., 1960.

The combinatorial problems of the eight queens in chess and the construction of orthogonal Latin squares are considered from the point of view of algorithms suitable for a digital computer.

An application of SWAC to the Queens problem verified the solutions for $n \leq 12$ as given by Kraitchik [*La mathématiques des jeux*, Brussels, 1930; p. 316] and found the number of solutions for $n = 13$.

A. D. Booth (London)

12049:

Mitrinović, Dragoslav S. Sur une relation de récurrence relative aux nombres de Bernoulli d'ordre supérieur. C. R. Acad. Sci. Paris **250** (1960), 4266–4267.

It is well known [Nörlund, *Differenzenrechnung*, Berlin, 1924; p. 148] that the Stirling number of the first kind S_n^r and the Bernoulli numbers $B_k^{(n)}$ of order n are related by means of

$$(1) \qquad S_n^{n-k} = \binom{n-1}{k} B_k^{(n)} \qquad (n \geq k-1).$$

The recurrence formula

$$(-1)^k S_n^{n-k} = \frac{1}{k}\binom{n}{k+1} \sum_{r=0}^{k-1} (-1)^r \frac{n-r}{r+1} \frac{\binom{k+1}{r}}{\binom{n}{r+1}} S_n^{n-r} \qquad (k < n)$$

was obtained in an earlier paper [D. S. Mitrinović and D. Djoković, C. R. Acad. Sci. Paris **250** (1960), 2110–2111; MR **22** #1522]. Using (1), this becomes

(2) $(-1)^k B_k^{(n)} = \frac{1}{k(k+1)} \sum_{r=0}^{k-1} (-1)^r (n-r) \binom{k+1}{r} B_r^{(n)}$.

A direct proof of (2), using the generating function, is given. Also explicit formulas for $B_{13}^{(n)}$, $B_{14}^{(n)}$, $B_{15}^{(n)}$ as polynomials in n are stated. *L. Carlitz* (Durham, N.C.)

12050:

Minc, Henryk. Mutability of bifurcating root-trees. Quart. J. Math. Oxford Ser. (2) **11** (1960), 187–192.

Compare the author's paper in Proc. Edinburgh Math. Soc. (2) **11** (1958/59), 207–209 [MR **22** #11019]. The number of bifurcating root-trees P which coalesce into the same tree when one ignores the distinction between right and left is of the form 2^μ, where $\mu = \mu(P)$ is an integer called the mutability of any of the trees P or of the corresponding nonassociative powers or indices. The reviewer [Proc. Roy. Soc. Edinburgh **59** (1939), 153–162] found certain inequalities connecting μ with the altitude α and degree (or in the author's terminology, potency) δ of P, amounting to maximal conditions for μ. The author finds two such inequalities which are minimal conditions for μ and shows that they are best possible.

I. M. H. Etherington (Edinburgh)

12051:

Wagner, K. Bemerkungen zu Hadwigers Vermutung. Math. Ann. **141** (1960), 433–451.

The paper is devoted to the study of the contraction of graphs by the process of identifying vertices connected by an edge repeatedly applied. The principal result is the complete characterisation of the structure of the graphs which are such that they cannot be contracted into a 5-clique with one edge missing, but the graph obtained by adding any new edge can be. It is shown that all these graphs are 4-colourable. *G. A. Dirac* (Hamburg)

12052:

Dirac, Gabriel Andrew. 4-chrome Graphen und vollständige 4-Graphen. Math. Nachr. **22** (1960), 51–60.

A graph is n-colorable if its points can be colored with n colors so that no two adjacent points have the same color. An n-chromatic graph G is n-colorable but not $(n-1)$-colorable. A graph G is n-critical if it is n-chromatic but $G-v$ is $(n-1)$-colorable for any point v. Let K_4 be the complete graph with 4 points and let H_4 be any graph homeomorphic to K_4. This paper studies properties of a 4-critical graph G.

Typical theorems: If u and v are points of G joined by three disjoint paths, then G has a subgraph H_4 containing these paths and having u and v as vertices (Satz 1). If v is a point of a cycle z of G, then G has an H_4 containing z and having v as a vertex (Satz 2). Any three lines of G lie in a subgraph H_4 (Satz 5).

F. Harary (Ann Arbor, Mich.)

12053:

Dirac, Gabriel Andrew. In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterteilungen. Math. Nachr. **22** (1960), 61–85.

A graph is n-connected if the removal of any $n-1$ points results in a connected graph with more than one point. For the remaining definitions, see the preceding review. Several interesting combinatorial results are obtained, including the following. Theorem 1: If G contains no H_4, then G is planar. Theorem 2: If the join of G with one new point is planar, then G contains no H_4. Theorem 5: If G is a graph in which all but at most one point have degree ≥ 3, then G contains an H_4. Theorem 6: If G has p points and q lines, $p \geq 4$, $q \geq 2p-2$, then G contains an H_4. Theorem 7: If G is a 2-connected graph in which each point has degree ≥ 3, then any point and any line of G lie in an H_4 of G. Theorem 9: If G is n-connected, $n \geq 2$, then any 2 lines and any $n-2$ points of G lie in a cycle. Theorem 12: Let z be a cycle of length k in a 3-connected graph G with p points. Then the number of distinct H_4's in G containing z is at least $\max\{p-k,\ k/4\}$. Theorem 15: Let v be a point of a cycle z of an n-connected graph G, $n \geq 3$. Then G has at least $(n-2)^2$ subgraphs H_4 containing z and having v as a vertex. Theorem 20: Any $n+2$ points of an n-connected graph G with $n \geq 4$ and p points, $p \geq n+2$, lie in an H_4 of G. This does not hold for $n=3$. The author concludes by stating that he does not know whether there exists a 4-critical, 4-connected graph.

F. Harary (Ann Arbor, Mich.)

12054:

Mayeda, W. Synthesis of switching functions by linear graph theory. IBM J. Res. Develop. **4** (1960), 321–328.

For the purpose of application to the synthesis of switching (boolean) functions, a matrix criterion is provided for determining whether a given binary matrix is a fundamental cut-set matrix of a graph. The accompanying examples facilitate the comprehension of the article. *F. Harary* (Ann Arbor, Mich.)

12055a:

Ádám, András. Über zweipolige elektrische Netze. I. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 211–218. (Hungarian. Russian and German summaries)

12055b:

Ádám, András. Über zweipolige elektrische Netze. II. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 67–79. (Hungarian. Russian and German summaries)

12055c:

Pollák, György. Bemerkung zur Arbeit "Über zweipolige elektrische Netze. II." von A. Ádám. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 81–82. (Hungarian. Russian and German summaries)

12055d:

Ádám, András. Über zweipolige elektrische Netze. III. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 207–218. (Hungarian. Russian and German summaries)

A graph (1-complex) with two distinguished vertices may represent an electric circuit, which suggests rules of composition by connecting in parallel and in series. These operations are studied. Applications to truth-functions are given in III. Sample theorem: A graph can be obtained by series and parallel connections from single edges if and only if no part of it is like the Wheatstone Bridge. *P. Ungar* (New York)

ORDER, LATTICES

See also 12023, 12135, 12507, 12508.

12056:

Močul'skiĭ, E. N. On certain classes of solvable lattices. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 283–316. (Russian)

12057:

Jónsson, Bjarni. Lattice-theoretic approach to projective and affine geometry. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 188–203. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959.

This is a very clear and fairly short exposition, without proofs, of the relationship between lattices and geometries, especially projective and affine geometries. There is an extensive bibliography, which, however, is not keyed to the text. *P. M. Whitman* (Silver Spring, Md.)

12058:

Naito, Tadao. Lattices with P-ideal topologies. Tôhoku Math. J. (2) **12** (1960), 235–262.

The author defines a P-ideal topology of a lattice L to be any topology of L which has a sub-basis of closed sets consisting of prime ideals and prime dual ideals. Two particular P-ideal topologies come in for special attention: the MP-ideal topology obtained by taking as a sub-basis all prime ideals and prime dual ideals which are minimal with respect to containing some element of L; the CP-ideal topology obtained by taking as a sub-basis all complete prime ideals and complete prime dual ideals. For example, the MP-ideal topology and CP-ideal topology on any chain coincide with the ordinary interval topology. It is shown that the lattice operations are continuous in any P-ideal topology, but order-convergent sequences need not converge in P-ideal topologies. The separation axioms for the P-ideal topologies of a lattice L impose strong restrictions on L. For example, if L is not distributive, then it admits no T_0 P-ideal topology. Conversely, if L is distributive, then its MP-ideal topology is T_1. However, the CP-ideal topology is not T_1 unless L is infinitely distributive, and even this is not sufficient. A major part of the paper is devoted to the study of the CP-ideal topology on infinitely distributive lattices, and in particular, on commutative l-groups. A few sample results of this study are: conditional completeness implies compactness of bounded subsets; group operations are continuous; in a conditionally complete l-group, all separation axioms are equivalent.

R. S. Pierce (Seattle, Wash.)

12059:

Skornyakov, L. A. On a lattice isomorphism of modules over regular rings. Dokl. Akad. Nauk SSSR **131** (1960), 756–757 (Russian); translated as Soviet Math. Dokl. **1**, 339–340.

Let F^n be a free unitary module with n generators over a regular ring F. Submodules of the module F^n which possess a finite number of generators form a complemented modular lattice $\mathfrak{C}(F^n)$. The author states, without proof, that the following theorem can be obtained using methods similar to that of Baer [*Linear algebra and projective geometry*, Academic Press, New York, 1952; MR **14**, 675]. Theorem: If F and G are regular rings, $n \geq 3$, and the lattice $\mathfrak{C}(F^n)$ is complete and continuous, then any isomorphism of the lattice $\mathfrak{C}(F^n)$ on the lattice $\mathfrak{C}(G^n)$ is induced by a semi-linear transformation of the module F^n on the module G^n. Related theorems are also given.

F. Maeda (Hiroshima)

12060:

Prather, Ronald. Computational aids for determining the minimal form of a truth function. J. Assoc. Comput. Mach. **7** (1960), 299–310.

The author gives but another computational aid for calculating prime implicants and essential prime implicants, this one by coding Boolean functions into pairs of decimal integers. Nothing is done toward resolving the main difficulty, namely, that of determining the minimal form from the prime implicants, other than by trial and error. *S. Ginsburg* (Santa Monica, Calif.)

12061:

Žuravlev, Yu. I. Separability of subsets of vertices of the n-dimensional unit cube. Trudy Mat. Inst. Steklov. **51** (1958), 143–157. (Russian)

The author gives a geometrical approach to the problem of finding a minimum normal form for a Boolean function with "don't-care" conditions. The domain of such a function may be considered as a certain subset of the vertices of an n-dimensional cube, its values 0 or 1. The problem then reduces to finding a minimum cover of the vertices by certain subsets of the cube [cf. Urbano and Mueller, Trans. IRE **EC-5** (1956), 126–132]. The author's algorithms seem to be similar to Quine's and to the first step of the extraction algorithm of the reviewer [Proc. Internat. Sympos. Switching Theory, 1957, Part I, 57–73, Harvard Univ. Press, Cambridge, Mass., 1959; MR **22** #5512]. *J. P. Roth* (Yorktown Heights, N.Y.)

12062:

Vasil'ev, Yu. L. Comparison of the complexity of impasse d.n.f. and minimal d.n.f. Dokl. Akad. Nauk SSSR **137** (1961), 13–16 (Russian); translated as Soviet Math. Dokl. **2**, 214–217.

A Boolean function g absorbs a function h if $g \cdot h = h$. A conjunction X absorbed by a function f is called minimal with respect to f if the conjunction X_i, the result of canceling the variable x_i in X, is not absorbed by f (for all x_i in X). A disjunctive normal form (d.n.f.) T is called an impasse d.n.f. if: (a) all conjunctions K in T are minimal with respect to the function f expressed by T; (b) each conjunction occurs in T only once and no conjunction of T is absorbed by the disjunction of the remaining conjunctions of T. The author introduces a measure of the "complexity" of a d.n.f. and shows that there is an impasse d.n.f. which is appreciably more complex than the minimal d.n.f.'s for the same function. On the other hand, there are certain classes of functions

for which arbitrary impasse d.n.f.'s are not "much more complex" than the minimal d.n.f.'s.

E. Mendelson (New York)

12063:

Halmos, Paul R. The representation of monadic Boolean algebras. Duke Math. J. **26** (1959), 447–454.

The main purpose here is to present new proofs for the main theorems of the first paper in the sequence on monadic algebras and polyadic algebras [Compositio Math. **12** (1956), 217–249; MR **19**, 112]. This paper incidentally furnishes a very good summary of the highlights of that paper, and could very profitably be read first (and again afterwards). The theorems involved are the representation theorems of monadic algebras, and the theorems on the existence of "constants". Using the dual spaces of monadic algebras, the author also points out some interesting relationships among three topological theorems due to Sikorski [Ann. Polon. Math. **21** (1948), 332–335; MR **11**, 76], Gleason [#12509] and Michael [Amer. Math. Monthly **63** (1956), 233–238], and the representation theorem mentioned above.

B. A. Galler (Ann Arbor, Mich.)

GENERAL MATHEMATICAL SYSTEMS

12064:

O'Keefe, Edward S. On the independence of primal algebras. Math. Z. **73** (1960), 79–94.

Author's summary: "The algebraic structure common to Boolean and Post algebras and to p-rings has recently been extended by A. L. Foster to a comprehensive structure theory for universal algebras [same Z. **58** (1953), 306–336; **59** (1953), 191–199; **62** (1955), 171–188; **65** (1956), 70–75; **66** (1957), 452–469; MR **15**, 194; **17**, 452; **18**, 108, 788]. In the most general phase of this latter theory the concept 'primal cluster' (of arbitrary species Sp) plays a central role. Each such cluster determines a unique sub-direct factorization for each algebra equationally embedded in the cluster, i.e., an algebra satisfying all identities common to some finite subset of the cluster. Various categories of primal clusters are known, largely clusters of species [2, 1].

"The principal objective of this paper is to establish new categories of primal clusters, thus broadening the domain of application of the general structure theory. It is shown, in particular, that the totality of pairwise non-isomorphic primal algebras of species $[s]$ constitute a primal cluster. Furthermore, the relationship between pairwise independence and independence for general algebras is clarified."

Ph. Dwinger (Lafayette, Ind.)

12065:

Čupona, Ǵ. On the ternary associative operations. Bull. Soc. Math. Phys. Macédoine **9** (1958), 5–10. (Macedonian. English summary)

Let A and B be any n-tuple and m-tuple operations in a set M respectively; this means in particular that, for every n-sequence x of elements of M, Ax means a well-defined member of M. A is primitive with respect to B, or B is induced by A, provided $m=1$ and $Bx=x$ $(x\in M)$, or $m>1$ and there exists an n-sequence $B_1, \cdots, B_n$ of m_i-tuple operations $(i=1, \cdots, n)$ induced by A such that, for every $x_1, \cdots, x_m$, $B(x_1\cdots x_m)=A[B_1\cdots B_i\cdots B_n(x_1\cdots x_{m_n})]$. A sequence $(e_1\cdots e_{n-1})$ is i-neutral provided that, for all $x\in M$,

$$A(e_1\cdots e_{i-1}xe_i\cdots e_{n-1})=x;$$

an $e\in M$ is i-neutral provided the constant sequence $(ee\cdots e)$ is i-neutral; an $e\in M$ is neutral for A, provided e is i-neutral for every $i\leqq n$. Using preceding notions the author proves six theorems; the main theorem concerns ternary operations and reads as follows (Theorem 1.1): Let A be associative and let (e_1e_2) be 2-neutral. Then (1) (e_1e_2) and (e_2e_1) are i-neutral for $i=1, 2, 3$; (2) for all $x, y\in M$, $A(e_ixy)=A(xe_iy)=A(xye_i)$ for $i=1, 2$; (3) (f_1f_2) is 1-neutral if and only if (f_1f_2) is 3-neutral; and (4) $A(xe_1^{2k})=y$ if and only if $x=A(ye_2^{2k})$ (the meaning of e_1^{2k} is obvious).

Đ. Kurepa (Zagreb)

CLASSICAL ALGEBRA

12066:

Chrystal, G. ★Algebra: An elementary text-book for the higher classes of secondary schools and for colleges. 6th ed. Chelsea Publishing Co., New York, 1959. Part I: xxiv+571 pp. Part II: xxiv+616 pp. $2.95 each.

A reprinting of the sixth edition.

12067:

Varoli, Giuseppe. Sulla ricerca delle radici di una equazione algebrica a coefficienti razionali. Period. Mat. (4) **37** (1959), 147–155.

Let the equation $a_0x^n+\cdots+a_n=0$ $(n\geqq 1, a_0\neq 0)$ have rational integral coefficients. The author generalizes the well-known necessary condition in order that a rational number be a root of such an equation by showing that a number of the form $(a+bi)/c$, where a, b, c are rational integers with $(a, b, c)=1$, can be a root only if c divides a_0 and a^2+b^2 divides ca_n. He also proves some simple results which can be used to decrease the number of cases to be considered when rational roots, or roots of the form just given, are found by trial.

H. W. Brinkmann (Swarthmore, Pa.)

12068:

Vyhandu, L. K. [Võhandu, L.]. On factorizing of polynomials. Tartu Riikl. Ül. Toimetised **73** (1959), 146–156. (Russian. Estonian and English summaries)

A rather disjointed account, which first presents a particular scheme for carrying out Bairstow's method by utilizing a special type of vector product of two vectors. The second part of the paper proposes a combination of two other known methods for obtaining a quadratic factor.

A. S. Householder (Oak Ridge, Tenn.)

12069:

Magula, D. A. A conjugate factor method for the solution of a cubic. Math. Comp. **14** (1960), 281–283.

Author's introduction: "This paper gives a simple method for computing the real roots of the reduced cubic

equation with real coefficients, $x^3+Ax+B=0$, having roots a, b, c. We assume a to be real, since every cubic equation has at least one real root.

"The method consists in factoring B, and setting one factor equal to $\pm\sqrt{m}$, the other n. For all pairs m, n such that $m+n=-A$, $\pm\sqrt{m}$ is a root. If no such pair exists, a method of interpolation is shown."

THEORY OF NUMBERS

See also 12223, 12244, B12579, B12739.

12070:

Delcourte, M. Indicateur d'Euler et nombres de Fermat. Mathesis **68** (1959), 350–356.

In this note the author establishes a number of very elementary propositions concerning Euler's ϕ-function. He gives in particular a proof of the following result due to R. D. Carmichael [Bull. Amer. Math. Soc. **13** (1907), 241–243]: The equation $\phi(x)=2^n$ has just $n+2$ solutions (a single one being odd) when $n\leq 31$ and just 32 solutions (not 33 as stated by Carmichael) when $32\leq n\leq 255$. The argument is based on the fact that the Fermat number $2^{2^n}+1$ is prime for $n=0$, 1, 2, 3, 4 and composite for $n=5$, 6, 7. *A. L. Whiteman* (Princeton, N.J.)

12071:

Gloden, A. A note on factors of n^4+1. Math. Comp. **14** (1960), 278–279.

New factorization of n^4+1 for 97 values of n up to 3000. Only 15 contain two large prime factors. [See Chiffres **2** (1959), 209–218; MR **22** #2028.]

D. H. Lehmer (Berkeley, Calif.)

12072:

Osborn, Roger. ★Tables of all primitive roots of odd primes less than 1000. University of Texas Press, Austin, Tex., 1961. 70 pp. $3.00.

The main table of this work gives all primitive roots g for $0<g<p$ for each odd prime $p<1000$. A previous similar table in the Italian edition of Chebyshev's *Teoria delle congruenze* extends only to $p<353$. Most tables give only one primitive root. A small table (pp. 69–70) gives data on primitive roots as follows. For each of the 168 primes is given the number of primitive roots $<p/2$, the number P of primitive roots prime to $p-1$ and the expected number $\{\phi(p-1)\}^2/(p-1)$. The author conjectures that $P>0$. A table showing the number of primes $p<1000$ having a given positive or negative number g as a primitive root would have been of interest.

D. H. Lehmer (Berkeley, Calif.)

12073:

Kravitz, Sidney. The congruence $2^{p-1}\equiv 1 \pmod{p^2}$ for $p<100{,}000$. Math. Comp. **14** (1960), 378.

The author announces the computation of the residues of $2^{p-1} \pmod{p^2}$ for $50{,}000<p<100{,}000$ [for values up to 50,000 see Fröberg, Math. Tables Aids Comput. **12** (1958), 281]. No residue congruent to 1 was found.

O. Taussky-Todd (Pasadena, Calif.)

12074:

Johnson, S. M. A linear diophantine problem. Canad. J. Math. **12** (1960), 390–398.

Let $a_1, a_2, \cdots, a_k$ be positive relatively prime integers. Several authors [see, for instance, J. B. Roberts, Proc. Amer. Math. Soc. **7** (1956), 465–469; Canad. J. Math. **9** (1957), 219–222; MR **19**, 1038] obtained bounds $F(a_1, a_2, \cdots, a_k)$ such that the equation $a_1x_1+a_2x_2+\cdots+a_kx_k=n$ has solutions in positive integers $x_1, x_2, \cdots, x_k$ for every integer $n>F(a_1, a_2, \cdots, a_k)$. For $k=2$ the best bound is $a_1\cdot a_2$. The author studies the case $k=3$. He reduces the problem to the case that a_1, a_2, a_3 are relatively prime in pairs. He develops a method to compute the best possible bound $F(a_1, a_2, a_3)$. He states without a proof that some of his results hold for the case $k>3$. *A. Brauer* (Chapel Hill, N.C.)

12075:

Yaakson, H. On symmetric solutions of a Diophantine equation. Uč. Zap. Tartu Gos. Univ. **46** (1957), 63–84. (Russian. Estonian summary)

12076:

Vinogradov, A. I. On an estimate for quadratic forms used in arithmetics. Vestnik Leningrad. Univ. **14** (1959), no. 19, 60–63. (Russian. English summary)

An estimate is given for the sum $\sum_{x=1}^{P}\tau^4(C+Bx-Ax^2)$, where $\tau(y)$ denotes the number of divisors of y and P, A, B, C and $D=B^2+4AC$ are integers for which constants N and $\alpha>0$ exist such that $|D|<N$ and $P>N^{\alpha}$.

W. H. Simons (Vancouver, B.C.)

12077:

Carlitz, Leonard. Some arithmetic properties of the lemniscate coefficients. Math. Nachr. **22** (1960), 237–249.

The coefficients in question are integers defined by the equations

$$\Phi(x)=\sum_{n=0}^{\infty}a_{4n+1}\frac{x^{4n+1}}{(4n+1)!},\quad a_1=1,\quad [\Phi'(x)]^2=1-[\Phi(x)]^4.$$

The author derives a number of congruence relations for these integers, modulo powers of 2. For example, he shows that $a_{4n+1}=2^{2n}c_n$, where the c_n are odd integers. He also studies the coefficients generated by $x/\Phi(x)$ and $x^2/\Phi^2(x)$.

T. M. Apostol (Pasadena, Calif.)

12078:

Shapiro, Harold N. On a theorem concerning exponential polynomials. Comm. Pure Appl. Math. **12** (1959), 487–500.

Basing his work on previous results by Skolem [Skr. Norske Vid. Akad. Oslo no. 6 (1933)], Mahler [Akad. Wetensch. Amsterdam Proc. **38** (1935), 50–60], and Lech [Ark. Mat. **2** (1953), 417–421; MR **15**, 104], the author studies in more detail the set of integers in which a function

$$F(x)=\sum_{j=1}^{m}A_j^xP_j(x)$$

can vanish; here $A_1, \cdots, A_m$ are distinct numbers, and $P_1(x), \cdots, P_m(x)$ are polynomials with coefficients in a field of characteristic 0. His most interesting result is that $F(x)$ can be written as a product of two functions of

the same kind where the first puts the zeros of $F(x)$ in evidence. [See also J. F. Ritt, Trans. Amer. Math. Soc. **31** (1929), 654–679, 680–686; and K. Mahler, Proc. Cambridge Philos. Soc. **52** (1956), 39–48; MR **17**, 597.]

K. Mahler (Manchester)

12079:

McKean, H. P., Jr. A problem about prime numbers and the random walk. I. Illinois J. Math. **5** (1961), 351.

It is shown that with probability one the 3-dimensional random walk hits the x-axis in a point the abscissa of which is a prime number infinitely often.

A. Rényi (Budapest)

12080:

Erdős, P. A problem about prime numbers and the random walk. II. Illinois J. Math. **5** (1961), 352–353.

Let $e(n)$ be equal to the number of primes $p \leq n$ such that the 3-dimensional random walk passes through the point with abscissa p on the x-axis. It is shown that $e(n) \cdot (\log \log n)^{-1}$ tends in probability to a positive constant c. This implies that with probability one the random walk hits an infinity of points on the x-axis the abscissa of which is a prime. It is indicated that by the same method it can also be proved that $e(n) \cdot (\log \log n)^{-1}$ tends to c with probability one. *A. Rényi* (Budapest)

12081:

Buhštab, A. A. An asymptotic estimate of certain numerical functions connected with the number of divisors. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 45–53. (Russian)

12082:

Capstick, W. A. On the Fibonacci sequence. Math. Gaz. **41** (1957), 120–121.

12083:

Statulevičius, V. On the representation of odd numbers as the sum of three almost equal prime numbers. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **3** (1955), 5–23. (Lithuanian. Russian summary)

Author's summary: "The following theorem is proved. The number $I(N, K)$ of solutions of the equation $N = p_1 + p_2 + p_3$, where N is an odd positive number, in primes p_1, p_2, p_3 subject to the condition $\frac{1}{2}N - K \leq p_i \leq \frac{1}{2}N + K$ $(i = 1, 2, 3)$, with $\frac{1}{2}N - 2 \geqq K \geqq N^{279/308+\varepsilon}$, and $\varepsilon > 0$ an arbitrarily small number, is expressed by the asymptotic formula

$$I(N, K) = \frac{3K^2}{\ln^3 N} S(N) + O\left(\frac{K^2}{\ln^4 N}\right),$$

where

$$S(N) = \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p \mid N} \left(1 - \frac{1}{p^2 - 3p + 3}\right) > \frac{3}{5},$$

and the constant in the symbol O does not depend on N and K and is effectively calculated.

"Thus, every sufficiently large odd number can be represented as a sum of three primes satisfying the condition $|p - \frac{1}{2}N| \leq N^{279/308+\varepsilon}$."

12084:

Gordon, Basil; Rubel, L. A. On the density of sets of integers possessing additive bases. Illinois J. Math. **4** (1960), 367–369.

The subject is the relation between an infinite set of positive integers $K = \{k_0, k_1, k_2, \cdots\}$ and the set S of integers which can be expressed as the sum of distinct elements of K. If $c_n = 1$ or 0 according as n is or is not in S, then $\sum_{n=0}^{\infty} c_n x^n = \prod_{n=0}^{\infty} (1 + x^{k_n})$. It was proved by Wintner [Rev. Un. Mat. Argentina **13** (1948), 99–105; MR **10**, 432] that if $\lim_{n \to \infty} 2^n / k_n = \theta$, then $\lim_{n \to \infty} (c_0 + \cdots + c_n)/n = \theta$. Here the converse is proved on the assumption that $\theta > 0$. The case $\theta = 0$ is undecided. Some examples are given in which there is no density θ in the above sense. *E. C. Titchmarsh* (Oxford)

12085:

Erdős, P. Remarks on number theory. III. On addition chains. Acta Arith. **6** (1960), 77–81.

[For part II see same Acta **5** (1959), 171–177; MR **21** #6348.] For a given natural number n, a sequence $1 = a_0 < a_1 < a_2 < \cdots < a_k = n$ of integers is called an addition chain if every a_l $(l \geqq 1)$ can be written as $a_l = a_i + a_j$ $(0 \leq i, j < l)$. Let $l(n)$ denote the number of elements in the shortest addition chain linking 1 and n. Then clearly $l(n) \geqq \log n / \log 2$ (equality occurs only if $n = 2^n$) and A. Brauer proved [Bull. Amer. Math. Soc. **45** (1939), 736–739; MR **1**, 40] that $l(n) < (\log n / \log 2) + (1 + o(1)) \times (\log n / \log \log n)$. The author shows that Brauer's result is best possible by proving that, for almost all n, $l(n) = (\log n / \log 2) + (1 + o(1))(\log n / \log \log n)$. The author also states, without proof, a number of related results. In an attached list of errata two misprints are noted, but there are several others. For instance, in the last line but one on p. 78, the inequality should read $k + 1 > (\log n / \log 2) + (u_2 + u_3)/100$. *H. Halberstam* (London)

12086:

Eda, Yoshikazu. Einige Sätze über λ-freie Zahlen. Osaka Math. J. **12** (1960), 39–60.

It is assumed that λ and s are integers, $\lambda \geqq 3$, $s \geqq 5$. A λ-free number is a natural number not divisible by the λth power of any integer greater than 1. The author obtains an asymptotic formula for the number of representations of a natural number as a sum of s squares of λ-free numbers. He also determines which sufficiently large natural numbers have such representations. {Correction: Delete the word 'Professor' before the reviewer's name.} *T. Estermann* (London)

12087:

Grosswald, E. Partitions into prime powers. Michigan Math. J. **7** (1960), 97–122.

This paper gives asymptotic formulae for the number of partitions $p(n, m; k)$ of the integer n into kth powers of primes p in the range $2 \leq p \leq m$. These results supersede asymptotic formulae due to Hardy and Ramanujan for the number of partitions of n into powers of primes. Several special ranges of the variables are considered: (i) m fixed; (ii) $m \to \infty$ with $m = O(n^{\varepsilon})$; (iii) $m^{k+1} < n \log m$; $m \neq O(n^{\varepsilon})$ for all ε; (iv) $n \log m \leq m^{k+1} < n \log n$; (v) $m^{k+1} > n \log n$. The form and accuracy of the

asymptotic formulae depends on the range of the variables. The results may be compared with the more general but less accurate results of Haselgrove and Temperley [Proc. Cambridge Philos. Soc. **50** (1954), 225–241; MR **16**, 17]. These results apply to the number of partitions of n into m parts. *C. B. Haselgrove* (Manchester)

12088:

Wright, E. M. Partitions into k parts. Math. Ann. **142** (1960/61), 311–316.

Mit Hilfe von erzeugenden Funktionen wird gezeigt: Die Anzahl $p_k(n)$ der Lösungen der Diophantischen Gleichung $n_1+\cdots+n_k=n$ $(1\leqq n_1\leqq\cdots\leqq n_k)$ erfüllt

$$p_k(n)=\sum_{j=1}^{k}\sum_{t=1}^{[k/j]}a(k,j,t;n)n^{t-1},$$

wobei $a(k, j, t; n)$ nur von k, j, t und der Restklasse von n mod j abhängt. Das verschärft einen einfacher bewiesenen Satz des Referenten [Math. Ann. **138** (1959), 356–362; MR **21** #7188; Satz 2]. *G. J. Rieger* (Munich)

12089:

LeVeque, W. J. On the frequency of small fractional parts in certain real sequences. II. Trans. Amer. Math. Soc. **94** (1960), 130–149.

This paper corrects and extends Theorem 1 of a previous paper under the same title [same Trans. **87** (1958), 237–261; MR **20** #2314]. The correction needed was already referred to in the review cited. The extension reads as follows. Let $\langle x\rangle$ be the distance between x and the integer nearest x and let $g(x)$ satisfy the following conditions: (a) $xg(x)$ is non-increasing and $0<xg(x)<\frac{1}{2}$, for $x\geqq 0$; (b) $xg(x)\to 0$ as $x\to\infty$; (c) $\sum_1^\infty g(k)=\infty$. Then for almost all x, the number of solutions $m\leqq n$ of the inequality $\langle mx\rangle<g(m)$ is asymptotic to $2\sum_1^n g(k)$. A conjecture is formulated which has since been proved by Erdös.

C. G. Lekkerkerker (Amsterdam)

12090:

LeVeque, W. J. On the frequency of small fractional parts in certain real sequences. III. J. Reine Angew. Math. **202** (1959), 215–220.

Theorem: Let $f(n)$ be a non-negative non-increasing function of the natural numbers n such that $f(1)\leqq\frac{1}{2}$ and such that $\sum f(n)$ diverges. Let a_k $(k=1, 2, 3, \cdots)$ be an increasing sequence of positive integers and suppose that

$$\sum_{k\leqq n}\frac{f(k)}{a_k}\sum_{l\leqq k}(a_k,a_l)=O\{(\sum_{k\leqq n}f(k))^{2-\eta}\}$$

for some constant $\eta>0$, as $n\to\infty$. Then for every sequence α_k of real numbers the number of solutions of

$$\|a_lx-\alpha_l\|\leqq f(l)\qquad(l\leqq k)\tag{1}$$

is $\{2+o(1)\}f(k)$ $(k\to\infty)$ for almost all real numbers x. Here $\|y\|$ denotes the distance of the real number y from the nearest integer.

This theorem thus gives a condition under which an analogue of the strong law of large numbers holds for the events (1). The analogue of the strong law of large numbers has been shown to hold in other cases, not all of them covered by this theorem [see #12089 and reference therein; see also #12091]. The proof of the theorem is particularly simple, consisting of little more than an estimate of variance and an application of the Borel-Cantelli lemmas. The cross-terms occurring in the variance are estimated by an application of trigonometric sums which may have wider application.

J. W. S. Cassels (Cambridge, England)

12091:

Erdös, P. Some results on diophantine approximation. Acta Arith. **5**, 359–369 (1959).

Let $\varphi(n, \varepsilon, C)$ be the set of $\alpha\in(0, 1)$ for which the inequality $|q\alpha-p|<\varepsilon q^{-1}$ has no integral solutions p and q with $n<q<Cn$ and $(p, q)=1$. The author, Szüsz and Turán conjectured [Colloq. Math. **6** (1958), 119–126; MR **21** #1290] that for every ε and C, $\lim_n \mathrm{meas}[\varphi(n, \varepsilon, C)]$ exists. This question remains open, but it is shown here that for every ε and η there is a $C=C(\varepsilon, \eta)$ such that for every n, $\mathrm{meas}[\varphi(n, \varepsilon, C)]<\eta$. The same kind of proof (depending in the end on Chebyshev's inequality from probability theory) also yields the following theorem: Let $l(n)>0$ be a nondecreasing function for which $\sum(l(n))^{-1}$ diverges, and denote by $N(l, \alpha, n)$ the number of integers m such that $1\leqq m\leqq n$ and $m\alpha-[m\alpha]<(l(m))^{-1}$. Then for almost all α, $N(l, \alpha, n)\sim\sum_1^n(l(m))^{-1}$ as $n\to\infty$. This extends a theorem of the reviewer [#12089] in which it was further required that $l(n)=o(n^{-1})$.

W. J. LeVeque (Ann Arbor, Mich.)

12092:

Cigler, Johann. Ziffernverteilung in ϑ-adischen Brüchen. Math. Z. **75** (1960/61), 8–13.

Let the transformation $T(x)$ of the interval $0\leqq x\leqq 1$ onto itself be defined by $T(x)=(\{\vartheta x\})$, where $\vartheta>1$ is an arbitrary fixed number and $(\{y\})$ denotes the fractional part of the non-negative number y. It has been proved by the reviewer [Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493; MR **20** #3843] that there exists a measure μ on the Borel sets of the interval $(0, 1)$ which is invariant with respect to T, equivalent with Lebesgue measure, and ergodic. The same result has been independently obtained by A. O. Gel'fond [Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 809–814; MR **22** #702], who gave also an explicit formula for μ: $\mu(B)=\int_B g(t)dt$, where $g(t)=c\cdot\sum_{t<t_k}\vartheta^{-k}$; here $t_k=T^k(1)$ for $k\geqq 1$, $t_0=1$, and the constant c is determined by the condition $\int_0^1 g(t)dt=1$. Starting from this explicit formula for the measure μ, the author gives a new and simple proof of the invariance of the measure μ with respect to T. A proof of the ergodicity of T is also given. Every real number α with $0\leqq\alpha\leqq 1$ has a representation in the form (*) $\alpha=\sum_{k=1}^\infty\lambda_k(\alpha)\cdot\vartheta^{-k}$, where $\lambda_k(\alpha)$ is a non-negative integer defined by $\lambda_k(\alpha)=[\vartheta T^k(\alpha)]$; here $[y]$ denotes the integral part of the number y. The author proves that if ϑ is an algebraic number such that the representation of the number 1 in the form (*) is finite, that is, $1=\sum_{k=1}^{n+1}a_k\cdot\vartheta^{-k}$, and if α is supposed to be a random variable distributed in $(0, 1)$ according to the measure μ (in which case the $\lambda_k(\alpha)$ are random variables too), then the n-tuples of consecutive "digits" $(\lambda_{\nu+1}(\alpha), \lambda_{\nu+2}(\alpha), \cdots, \lambda_{\nu+n}(\alpha))$, $\nu=0, 1, 2, \cdots$, form a homogeneous Markov chain. This may be considered as a generalization of the well-known fact that if ϑ is an integer (in which case $n=0$ and μ is Lebesgue measure) then the random variables $\lambda_k(\alpha)$ are independent and have the same distribution; or of the fact that for $\vartheta=\frac{1}{2}(1+\sqrt{5})$ (in which case $n=1$) the digits $\lambda_k(\alpha)$ form a Markov chain. *A. Rényi* (Budapest)

12093:

Carlitz, L. A generalization of Maillet's determinant and a bound for the first factor of the class number. Proc. Amer. Math. Soc. **12** (1961), 256–261.

Let p be a prime not less than 3. For $(r, p)=1$ define r' by means of $rr' \equiv 1 \pmod p$; also let $R(r)$ denote the least positive residue of $r \pmod p$. Maillet [see T. Muir, *Contributions to the history of determinants, 1900–1920*, Blackie, Glasgow, 1930] defined the determinant D_p of order $(p-1)/2$ by means of $D_p=|R(rs')|(r, s=1, \cdots, (p-1)/2)$. The author's generalization is given by $D_p^{(n)}=|p^n \bar{B}_n(rs'/p)|$, where $\bar{B}_n(x)$ is the Bernoulli function defined by $\bar{B}_n(x)=B_n(x)$ $(0 \leq x<1)$, $\bar{B}_n(x+1)=\bar{B}_n(x)$, and $B_n(x)$ is the Bernoulli polynomial of degree n. For $n=1$ the determinant $D_p^{(n)}$ reduces to $-\frac{1}{2}D_p$. Using the generalized Bernoulli numbers of Leopoldt [Abh. Math. Sem. Univ. Hamburg **22** (1958), 131–140; MR **19**, 1161] the author proves that $D_p^{(n)} \neq 0$ $(n=1, 2, \cdots)$, and moreover that $D_p^{(n)}$ is divisible by at least $p^{(p-3)/2}$ for all $n \geq 1$. He also derives the following upper bound for h, the first factor of the class number of the cyclotomic field $R(e^{2\pi i/p})$: $h \leq (m-1)!$ or $(m-1)!m^{1/2}$ according as $p=4m+1$ or $p=4m+3$, respectively.

A. L. Whiteman (Los Angeles, Calif.)

12094:

Yamamoto, Koichi. Arithmetic linear transformations in an algebraic number field. Mem. Fac. Sci. Kyushu Univ. Ser. A Math. **12** (1958), 41–66.

Let k be an algebraic number field of absolute degree n, A_k the family of arithmetic functions $\alpha=\alpha(\mathfrak{a})$ defined on all integral divisors $\mathfrak{a}$ of the field k, F the family of complex-valued functions defined on $(0, \infty)$ and vanishing in $(0, 1)$. The author studies the operators in A_k and F: $(S_\alpha f)(x)=\sum_{\mathfrak{a}} \alpha(\mathfrak{a}) f(x/N\mathfrak{a})$, where $\mathfrak{a}$ runs through all integral divisors, $\alpha \in A_k$, $f(x) \in F$; $(\alpha * \beta)(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}} \alpha(\mathfrak{b})\beta(\mathfrak{a}/\mathfrak{b})$, $\alpha, \beta \in A_k$; and $T(\alpha)=\sum_{N\mathfrak{a} \leq x} \alpha(\mathfrak{a})$, $\alpha \in A_k$.

The family A_k forms a commutative ring under the operations of addition and the $*$-convolution. On the basis of these definitions the author gives a variant of a method of Selberg for the field k. Stating without proof the well-known Dedekind formula

$$\sum_{N\mathfrak{a} \leq x} 1 = gx+O(x^{1-\nu}), \tag{1}$$

where g is the density of the divisors of k, and $\nu=n^{-1}$, the author derives from (1) the following asymptotic formulas:

$$\begin{aligned} T(L^m) &= g \sum_{j=0}^{m} (-1)^j m(m-1)\cdots(m-j+1) \\ &\qquad \times x \lg^{m-j} x + O(x^{1-\nu} \lg^m x), \\ (S_{L^m})(x) &= g(m+1)^{-1} x \lg^{m+1} x \\ &\qquad + B_{m+1} x + O(x^{1-\nu} \lg^{m+1} x), \end{aligned} \tag{2}$$

where $L=\lg N\mathfrak{a}$ and where B_{m+1} is a constant depending on k and m.

The formulas (2) have, as a consequence, a number of new asymptotic identities, e.g.,

$$\sum_{\mathfrak{a}} (x/N\mathfrak{a})^{1-\nu} \lg^m x/N\mathfrak{a} = gn^{m+1} m! x + O(x^{1-\nu} \lg^{m+1} x),$$

$$\sum_{N\mathfrak{a} \leq x} \Lambda_m(\mathfrak{a}) = mx \lg^{m-1} x + O(x \lg^{m-2} x), \qquad m \geq 2,$$

where $\Lambda_m=\mu * L^m$ and $\mu=\mu(\mathfrak{a})$ is the Möbius function for k.

Let $\mathfrak{m}$ be an integral divisor, E an arbitrary residue class mod $\mathfrak{m}$ of the group of divisors of k, h the number of distinct classes E and $N(E, x)$ the number of those divisors of E for which $N\mathfrak{a} \leq x$. The author assumes without proof the well-known asymptotic formula of Weber-Dedekind

$$N(E, x) = gh^{-1}x+O(x^{1-\nu}) \tag{3}$$

and derives, by means of a method developed by him, a number of asymptotic identities containing the characters of the complete group of residue classes mod $\mathfrak{m}$; e.g.,

$$\sum \lg N\mathfrak{p}/N\mathfrak{p} = h^{-1} \lg x+O(1), \tag{4}$$

where the summation is extended over all those primes $\mathfrak{p}$ of E having absolute degree 1, and $N\mathfrak{p} \leq x$.

The identity (4) has application in class field theory; let K be a finite extension of k such that the relative norm of an arbitrary integral divisor $\mathfrak{a}$ of K belongs to the unit residue class mod $\mathfrak{m}$ if $\mathfrak{a}$ and $\mathfrak{m}$ are relatively prime; $(K: k)=n_0$. It is then shown that $h \leq n_0$ (the second inequality of class field theory).

The author shows finally that, by taking the well-known theorem of Hecke-Takagi that $\sum_{\mathfrak{a}} \chi(\mathfrak{a})/N(\mathfrak{a}) \neq 0$ (where $\mathfrak{a}$ runs through all integral divisors of k) for an arbitrary non-principal character $\chi(\mathfrak{a})$ of k, its method of proof allows him to obtain a theorem on the asymptotically uniform distribution of prime divisors in all residue classes mod $\mathfrak{m}$, i.e., $\psi_E(x) \sim h^{-1}\psi(x)$ as $x \to \infty$, where $\psi_E(x)$ and $\psi(x)$ are the Chebyshev functions for k and E respectively.

N. G. Čudakov (RŽMat **1959** #8799)

12095:

Tamagawa, Tsuneo. On Hilbert's modular group. J. Math. Soc. Japan **11** (1959), 241–246.

The author determines a fundamental domain for Hilbert's modular group. The methods are quite arithmetical, and seem to be simpler than those of O. Hermann [Math. Z. **60** (1954), 148–155; MR **16**, 16].

Let $\mathfrak{k}$ be a totally real field of degree d, and $\alpha \to \alpha^{(1)}, \cdots, \alpha \to \alpha^{(d)}$ distinct isomorphisms of $\mathfrak{k}$ into the complex field. $\mathfrak{k}$ is then identified with a subset of the d-dimensional complex space C^d. Let P be the set of points $(\tau^{(1)}, \cdots, \tau^{(d)})$ in C^d such that $\operatorname{Im}(\tau^{(i)})>0$. Denote by $\mathfrak{o}$ the ring of integers of $\mathfrak{k}$. The Hilbert modular group Γ is the group of transformations

$$\tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta} = \left(\frac{\alpha^{(1)}\tau^{(1)}+\beta^{(1)}}{\gamma^{(1)}\tau^{(1)}+\delta^{(1)}}, \cdots, \frac{\alpha^{(d)}\tau^{(d)}+\beta^{(d)}}{\gamma^{(d)}\tau^{(d)}+\delta^{(d)}}\right),$$

where $\tau \in P$, $\alpha, \beta, \gamma, \delta \in \mathfrak{o}$, and $\alpha\delta-\beta\gamma=1$.

If $x \in C^d$, define $|x|=(|x^{(1)}|, \cdots, |x^{(d)}|)$; if the components are real, let $\|x\|$ be the Euclidean norm; if the components are real and positive, define $\log x=(\log x^{(1)}, \cdots, \log x^{(d)})$. Let E be the group of units and $\mathfrak{a}$ an ideal of $\mathfrak{k}$. A point $\tau=\xi+i\eta \in P$ is called $\mathfrak{a}$-reduced if: (a) $\|\xi+\alpha\| \geq \|\xi\|$, for all $\alpha \in \mathfrak{a}$; (b) $\|\log \eta+2 \log |\varepsilon|\| \geq \|\log \eta\|$, for all $\varepsilon \in E$.

Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_h$ be a set of representatives of the ideal classes of $\mathfrak{k}$ such that (i) each $\mathfrak{a}_i$ is integral, and (ii) $\mathfrak{a}_i$ has minimum norm among the integral ideals in its class. $\mathfrak{a}_1$ is assumed to represent the principal class.

Define F_i to be the set of all $\tau \in P$ such that (1) τ is $\mathfrak{a}_i^{-2}$-reduced, and (2) $\prod_{n=1}^{d} |\gamma^{(n)}\tau^{(n)}+\delta^{(n)}| \geq 1$ for all pairs $(\gamma, \delta) \neq (0, 0)$ with $\gamma \in \mathfrak{a}_i$, $\delta \in \mathfrak{a}_i^{-1}$. It is shown that the infinitely many inequalities which define F_i may be replaced by a finite number among them.

Let γ_i, δ_i be integers generating a_i. Then there exist α_i, $\beta_i \in a_i^{-1}$ such that $\alpha_i\delta_i - \beta_i\gamma_i = 1$. Let A_i denote the transformation $\tau \to (\alpha_i\tau + \beta_i)/(\gamma_i\tau + \delta_i)$. Theorem: The set $F = F_1 \cup A_2^{-1}F_2 \cup \cdots \cup A_h^{-1}F_h$ is a fundamental domain for the group Γ. *L. Greenberg* (Copenhagen)

12096:

Eggan, L. C. On Diophantine approximations. Trans. Amer. Math. Soc. **99** (1961), 102–117.

In the sequel θ denotes an irrational number. A classical result of Hurwitz states that there are infinitely many pairs of integers p, q with $q^2|\theta - p/q| < 5^{-1/2}$. A. V. Prasad [J. London Math. Soc. **23** (1948), 169–171; MR **10**, 513], among others, has shown that if only one solution (p, q) is demanded, the appropriate constant is less than $5^{-1/2}$ and is, in fact, $\frac{1}{2}(3-\sqrt{5})$.

In the first part of this paper the author generalizes Prasad's result as follows: "Let n be a positive integer and let $\theta_n = ((n^2+4)^{1/2} - n)/2$. For any positive integer m, let $c_m = \theta_n + n + P_{2m-1}/Q_{2m-1}$, where P_j/Q_j is the jth convergent to the continued fraction expansion $[0, n, n, \cdots]$ of θ_n. Then if $\theta = [a_0, a_1, a_2, \cdots]$ and if $a_j \geqq n$ for infinitely many values of j, there are at least m solutions in relatively prime integers p, q $(q > 0)$ to the inequality $q^2|\theta - p/q| \leqq 1/c_m$. Moreover, the constant c_m cannot be improved, since for $\theta = \theta_n$ there are exactly m solutions and equality is attained." For $n = 1$ this result is due to Prasad. For $n = 2$ it provides the second link in the Markoff chain.

In the second part of this paper the author applies his method to other results in the theory of diophantine approximations. W. T. Scott [Bull. Amer. Math. Soc. **46** (1940), 124–129; MR **1**, 203] has shown that the inequality $q^2|\theta - p/q| < 1$ has an infinity of solutions (p, q) if we restrict the fractions p/q to those chosen from any one of the following three classes: (a) p, q both odd; (b) p odd and q even; (c) p even and q odd. The author shows that the constant 1 on the right is the best possible even if only one solution is demanded instead of infinitely many; in a similar way he treats a related result due to R. M. Robinson [Duke Math. J. **7** (1940), 345–359; MR **2**, 149]. A. Khintchine [Math. Ann. **111** (1935), 631–637] proved that for any real number α and for $\sigma > 0$ the inequality $v|v\theta - u - \alpha| < 5^{-1/2} + \sigma$ has an infinity of solutions in integers (u, v) with $v > 0$. In the paper reviewed here it is shown that the constant $5^{-1/2}$ cannot be replaced by any smaller number, even if only one solution is demanded.

Moreover, the author also attacks with his method the problem of asymmetric approximations, introduced by B. Segre [Duke Math. J. **12** (1945), 337–365; MR **6**, 258] and treated later by a number of other authors.

In the last part of this paper the author extends the following theorem of J. F. Koksma [Simon Stevin **28** (1951), 199–202; MR **13**, 825]. Let $\sigma > 0$ and let $s \geqq 1$, a, b denote arbitrary integers. Then there exists an infinity of solutions of the inequality

$$q^2|\theta - p/q| < 5^{-1/2}(1+\sigma)s^2$$

such that $p \equiv a \pmod{s}$, $q \equiv b \pmod{s}$.

For the numbers θ equivalent to $\frac{1}{2}(5^{1/2}-1)$ the constant $5^{-1/2}$ on the right is the best possible. The author sharpens this result by disregarding the numbers θ equivalent to $\frac{1}{2}(5^{1/2}-1)$. *J. Popken* (Amsterdam)

12097:

Cigler, Johann. Asymptotische Verteilung reeller Zahlen mod 1. Monatsh. Math. **64** (1960), 201–225.

Let G be the additive group of reals (mod 1), $T = (c_{mn})$ a regular summation method. Let $\{x_k\}$ be a sequence of points in G such that $T\text{-lim } f(x_k) = \mu \cdot f$ exists for each $f \in C(G)$. Then μ is a bounded linear functional on $C(G)$; the corresponding regular Borel measure on G is called the T-distribution of $\{x_k\}$. Approximating f by a trigonometric polynomial, we have (Weyl) that $\{x_k\}$ has the T-distribution ν if and only if $T\text{-lim } e^{2\pi i q x_k} = \nu \cdot e^{2\pi i q x}$ $(q = 0, \pm 1, \cdots)$.

It is observed that, in view of the boundedness of $\{f(x_k)\}$, $\{x_k\}$ has a $(C, 1)$-distribution ν if and only if it has a (C, k)-distribution ν $(k \geqq 1)$, and this is true if and only if it has an A-distribution ν. Further, $(C, 1)$ and (M, n^r) are equivalent if $r > -1$. Here, (M, p_n) is defined by $c_{mn} = p_m/(p_0 + p_1 + \cdots + p_n)$ if $n \leqq m$, $c_{mn} = 0$ if $n > m$, where $p_n \geqq 0$, $p_0 > 0$, $\sum p_n = \infty$. Some typical results in this paper are the following.

(I) If $f(x)$ has a continuous derivative, $f(x) \to \infty$ monotonely, $f'(x) \to 0$, then $\{f(k) \pmod 1\}$ has the $(C, 1)$-distribution ν if and only if $\{f(g(k)) \pmod 1\}$ has the $(M, g'(n))$-distribution ν, where (say) $g(n) = n^s$ with $0 < s \leqq 1$. Note that the latter is equivalent with a $(C, 1)$-distribution ν.

(II) If $f(x)$ has a continuous second derivative, $f(x) \to \infty$ monotonely, $f'(x) \to 0$ monotonely, then the only $(C, 1)$-distribution which $\{f(k) \pmod 1\}$ can have is the uniform distribution on $(0, 1)$. {Note of the reviewer: By other methods one can show that if $f(n) \to \infty$, $\Delta f(n) \to 0$ monotonely, then $\{f(k) \pmod 1\}$ has no $(C, 1)$-distribution unless $n\Delta f(n) \to \infty$, in which case (Féjèr) it has the uniform $(C, 1)$-distribution.}

(III) Let $f'(t) \to 0$ monotonely, $tf'(t) \to \infty$. Then the double sequence $\{f(aj + bk) \pmod 1\}$ has the uniform distribution on b as its $(M, 1, 1)$-distribution, the latter with respect to the sequence of rectangles $1 \leqq j \leqq cn$, $1 \leqq k \leqq dn$ (a, b, c, d constants).

Finally, the author determines under appropriate conditions the upper and lower (M, p_n)-distribution functions of a sequence, generalizing a result of J. F. Koksma [*Diophantische Approximationen*, Springer, Berlin, 1936; p. 88]; special attention is given to the case that p_n is an arithmetic function like $\tau(n)$, $\sigma^a(n)$ and $r(n)$. *J. H. B. Kemperman* (Madison, Wis.)

12098:

Singh, V. N. Ramanujan's continued fraction and the Bauer-Muir transformation. Proc. Cambridge Philos. Soc. **57** (1961), 76–79.

The author attacks a continued fraction identity, originally stated by Ramanujan, by modifying a method developed by Perron [Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1953**, 21–56; MR **16**, 29] employing the Bauer-Muir theorem. A new version of this theorem recently obtained by Watson [Proc. Edinburgh Math. Soc. **11** (1958/59), 167–174; MR **22** #3799] is used. The identity of Ramanujan under consideration had previously been proved by Watson [Proc. Cambridge Philos. Soc. **31** (1935), 7–17] by other means. *W. J. Thron* (Boulder, Colo.)

12099:

Gur'yanov, I. N. On a criterion for the convergence of

continued fractions with arbitrary partial numerators. Uspehi Mat. Nauk **15** (1960), no. 2 (92), 173–180. (Russian)

The paper gives certain criteria for the convergence of a continued fraction (1) $K=[1, c_2, c_3, \cdots]$, in which the partial denominators are all 1's, and the partial numerators $1, c_2, c_3, \cdots$ are complex numbers that may be functions of one or more variables. The positive numbers $r_1, r_2, \cdots$ are assumed to satisfy the inequalities (2) $|1+c_n+c_{n+1}| \geq |c_n|r_{n-2}+|c_{n+1}|/r_n$, where for $n=1$ or $n=2$ the quantities c_1 and r_0 are to be replaced by 0. Scott and Wall [Trans. Amer. Math. Soc. **47** (1940), 155–172; MR **1**, 217] showed that if the functions c_n satisfy the inequalities (2) on a set E, then $|K| \leq 1+\sum_1^\infty r_1r_2\cdots r_n$, and the convergence of the latter series is sufficient for uniform convergence of the continued fraction. The author first proves the lemma that if $|c_n| \leq t_n$ on E for $n \geq 2$, and if t_k are real numbers such that the convergents to the continued fraction $[1, -t_2, -t_3, \cdots]$ have positive denominators H_n, then the numbers r_n defined by $r_n=t_{n+1}H_{n-1}/H_{n+1}$ satisfy the inequalities (2). Briefly, $H_{n-1}|1+c_n+c_{n+1}| \geq H_{n-1}\times(1-t_n-t_{n+1}) = t_n(H_{n-2}-H_{n-1})+H_{n+1} \geq |c_n|t_{n-1}H_{n-3}+H_{n+1}|c_{n+1}|/t_{n+1}$, if $|c_n| < t_n = (H_{n-1}-H_n)/H_{n-2}$. Next the substitution $\gamma_n=p_{n-1}/p_n$ changes $t_n=(p_{n-1}-p_n)/p_{n-2}$ to $t_n=(\gamma_n-1)/\gamma_{n-1}\gamma_n$ for $n \geq 2$. Whenever $\gamma_1>0$, the conditions $|c_n| \leq t_n=(\gamma_n-1)/\gamma_{n-1}\gamma_n$ on a set E imply that the continued fraction K in (1) converges uniformly on E. Furthermore, $|K| \leq 1+1/(L-1)$ if $\gamma_1>1$, and $|K| \leq 1+1/(L-\gamma_1^2)$ if $0<\gamma_1<1$, where $L=\gamma_1+\gamma_1/\sum_{n=2}^\infty(\gamma_2-1)\cdots\times(\gamma_n-1)$. The author proves only the case $0<\gamma_1<1$, since the case $\gamma_1>1$ was treated by Scott and Wall and the case $\gamma_1=1$ by Khovansky. Next, if $\gamma_k=\gamma>1$ for all k, the continued fraction $(K^{-1}-1)/c_2$ has the form (1), and both the estimate $|1-K^{-1}| \leq 1/L$ and the equivalent estimate $|K-L^2/(L^2-1)| \leq L/(L^2-1)$ hold, not only for $\gamma \geq 2$ when $L=\gamma$, but also for $1<\gamma \leq 2$ when $L=\gamma/(\gamma-1)$. A final theorem generalizes results of Van Vleck and Pringsheim: In the continued fraction $K=[1, c_2z, c_3z, \cdots]$ let $|c_nz| < (\gamma-1)/\gamma^2$. Then the conditions $0 \leq c_n \leq g$ are sufficient for the convergence of K in $|z| < (\gamma-1)/\gamma^2g$ to a regular analytic irrational function which equals the series corresponding to this fraction, and moreover, $|K-L^2/(L^2-1)| < L/(L^2-1)$, where $L=\gamma$ when $\gamma \geq 2$, but $L=\gamma/(\gamma-1)$ for $0<\gamma<1$.

J. S. Frame (E. Lansing, Mich.)

12100:

Groemer, Helmut. The number of lattice points on the boundary of a star body. Proc. Amer. Math. Soc. **11** (1960), 757–762.

Suppose that the set S of all points X in Euclidean n-space E_n, defined by $f(X) \leq 1$, is a bounded star body [cf. K. Mahler, Proc. Roy. Soc. London Ser. A **87** (1946), 151–187; MR **8**, 195]. Then there exists a least number k such that, for all X, Y in E_n,

$$f(X+Y) \leq k\{f(X)+f(Y)\},$$

and this k is known as the concavity coefficient of S. By the properties of the distance function $f(X)$ it follows that $k \geq 1$, and moreover, that $k=1$ if and only if S is convex. Define $A_n(k)$ as the greatest number with the property that there exists an n-dimensional star body S with concavity coefficient k and a lattice having $A_n(k)$ points on the boundary of S but none, other than the origin 0, in the interior of S. A classical result of Minkowski [*Geometrie der Zahlen*, Teubner, Leipzig, 1910; p. 79] is $A_n(1)=3^n-1$. His method has been adapted by the author to give the following estimates for $A_n(k)$. Theorem: Let $m_i=[2k]+i$ $(i=1, 2, \cdots, n)$ and put

$$P_n(k) = \inf m_i^n \sum_{d|m_i} \mu(d)d^{-n},$$

$$Q_n(k) = \sum_{i=1}^{\infty} \mu(i)\{(2[k/i]+1)^n-1\}.$$

Then $Q_n(k) \leq A_n(k) \leq P_n(k)$. In particular, it may be remarked that, since $\sum \mu(d)d^{-n}<1$, a star body S (of concavity coefficient k) having more than $m_1^n-1=[2k+1]^n-1$ lattice points on its boundary has at least one lattice point not 0 in its interior. Thus Minkowski's result 3^n-1 is valid not only for $k=1$, but for all $k<\frac{3}{2}$. For large k, the following estimates for $P_n(k)$ and $Q_n(k)$ are given:

$$P_n(k) = 2^n\zeta^{-1}(n)k^n+o(k^n),$$

$$Q_n(k) = 2^n\zeta^{-1}(n)k^n+o(k^n),$$

as $k \to \infty$. Thus $A_n(k)=2^n\zeta^{-1}(n)k^n+o(k^n)$ as $n \to \infty$.

J. J. H. Chalk (Toronto)

FIELDS

See also 12068.

12101:

Zaikina, N. G. Distribution of the numbers of an imaginary quadratic field composed of small prime ideals. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 261–272. (Russian)

12102:

Segre, B. Sistemi di equazioni nei campi di Galois. Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), pp. 66–80. Edizioni Cremonese, Rome, 1960.

In this lecture the author summarizes the content of three papers on the determination of the number of the solutions (in the algebraic closure of γ) of a system of algebraic equations over a Galois field γ [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 155–161, 303–311; **28** (1960), 271–277; MR **22** #6785, 6786]. Two different methods are employed. The first one (§ 1) is based on the consideration of a system of algebraic equations having the form (1) $\varphi_i(x_i)=0$, $f(x_1, \cdots, x_r)=0$, $i=1, 2, \cdots, r$ (φ_i is a polynomial of degree n_i) over an arbitrary field γ, and of the coefficients a of the polynomials $f_{i_1,\cdots,i_r}$ of degree n_i-1 in x_i, defined by the congruences $f_{i_1,\cdots,i_r}(x_1, \cdots, x_r) \equiv x_1^{i_1}\cdots x_r^{i_r} f(x_1, \cdots, x_r) \bmod (\varphi_1, \cdots, \varphi_r)$. If ρ is the rank of the square matrix, of order $N=n_1\cdots n_r$, composed by the a, then the number k of solutions of the system (1) is given by $k=N-\rho$. This result may be generalized to a system of algebraic equations (2) $f_1=0, \cdots, f_s=0$, in the $x_1, \cdots, x_r$; if γ is a Galois field, of order $q=p^t$ (p a prime), then, assuming $\varphi_i(x_i)=x_i^q-x_i$, the previous result gives the number of solutions of the equation $f_1=0$ [of the system $f_1=0, \cdots, f_s=0$]. The second method (§ 5) gives the number k only mod p. It is possible to construct a (wholly

determined) polynomial $P(a)$, in the coefficients a of the polynomials f_i and of degree $\leq q$ in each of the a, so that $P(a) \equiv k \bmod p$.

The author gives many interesting applications of his results to the so-called "Galois geometries" (geometries of finite Desarguesian spaces); in particular (§ 6) he investigates, and determines in many cases, the number of points of quartics in a Galois space of dimension three, and that of equi-anharmonic cubics in a Galois plane.

L. Lombardo-Radice (Rome)

12103:

Ikeda, Masatoshi. Zur Existenz eigentlicher galoisscher Körper beim Einbettungsproblem für galoissche Algebren. Abh. Math. Sem. Univ. Hamburg **24** (1960), 126–131.

This paper is related—though independent work—to a paper of S. P. Demuškin and I. R. Šafarevič [Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 823–840; MR **22** #1566]. The author considers the problem of imbedding a normal extension K/k with Galois group $\mathfrak{g}$ into a normal extension Ω/k with Galois group $\mathfrak{G}$ such that the kernel $\mathfrak{g}_0$ of the Galois mapping $\mathfrak{G} \to \mathfrak{g}$ is an abelian group. Hasse [Math. Nachr. **1** (1948), 40–61; MR **10**, 426] found necessary conditions for the possibility of such imbeddings. However, he admitted semisimple commutative algebras Ω/k for solutions, called improper solutions of the problem. The author shows that the existence of improper solutions implies the existence of proper solutions, i.e., for which Ω/k is a field. First it is shown that it suffices to treat abelian p-groups $\mathfrak{g}_0$; secondly, the proof of the theorem is made to depend on an earlier result of I. R. Šafarevič [Izv. Akad. Nauk SSSR Ser. Mat. **18** (1954), 389–418; MR **16**, 1001] concerning the imbedding of K/k into a field Ω' with Galois mapping on $\mathfrak{g}$ whose kernel is a sufficiently large direct product of $\mathfrak{g}_0$ by itself. Finally, the Hasse conditions [loc. cit.] on the triviality of certain obstructions are expressed in terms of factor sets (cyclic algebras) by assuming that the roots of unity whose order equals the exponent of $\mathfrak{g}_0$ lie in K [see also the paper of Demuškin and Šafarevič].

O. F. G. Schilling (Chicago, Ill.)

12104:

Gaschütz, Wolfgang. Fixkörper von p-Automorphismengruppen rein-transzendenter Körpererweiterungen von p-Charakteristik. Math. Z. **71** (1959), 466–468.

Soient $K = k(x_1, x_2, \cdots, x_n)$ une extension transcendente pure de degré de transcendence n d'un corps k et G un groupe fini de transformations linéaires à coefficients dans k des indéterminées $x_1, x_2, \cdots, x_n$. Il se pose le problème: le corps des invariants de G est-il aussi une extension transcendente pure de k? L'auteur donne la réponse positive à cette question quand la caractéristique p de k est $\neq 0$ et G est un p-groupe. Il démontre, dans ce but, le lemme plus général suivant: si $k = K_0 \subset K_1 \subset \cdots \subset K_n = K$ est une chaîne de corps de caractéristique $p \neq 0$ telle que $K_i = K_{i-1}(u_i)$, et si G est un p-groupe fini d'automorphismes de K/k tel que, pour tout $\sigma \in G$ et pour tout $i = 1, 2, \cdots, n$, on ait $u_i^\sigma - u_i \in K_{i-1}$, il existe n éléments $z_1, z_2, \cdots, z_n$ de K tels que le corps des invariants de G dans K soit $k(z_1, z_2, \cdots, z_n)$. Le résultat cité en résulte immédiatement, car un p-groupe de transformations linéaires sur un corps de caractéristique p peut se mettre sous une forme triangulaire par une transformation linéaire des indéterminées.

M. Krasner (Paris)

12105:

Posner, Edward C. Integral closure of differential rings. Pacific J. Math. **10** (1960), 1393–1396.

The main result is the following: Let F be a field of characteristic zero, $A = F[z_1, z_2, \cdots, z_n]$ a commutative finitely generated ring extension of F. Let D be a (finite or infinite) family of derivations of A into itself over F. Let A be differentiably simple under D. Then A is integrally closed in its quotient field K. A counter example for $n = 2$ shows that the converse does not hold in general. However, the following theorem is proved for the one-variable case. Let K be a function field in one variable over a field F of characteristic zero, and let x be an element of K transcendental over F. Let O_x denote the set of elements of K integral over x. Then O_x is differentiably simple with field of constants F under a family of two or fewer derivations.

H. Levi (New York)

ABSTRACT ALGEBRAIC GEOMETRY

See 12166, 12244, 12539, 12543.

LINEAR ALGEBRA

See also 12188, B12702.

12106:

Frazer, R. A.; Duncan, W. J.; Collar, A. R. ★Elementary matrices, and some applications to dynamics and differential equations. Cambridge University Press, New York, 1960. xviii + 416 pp. $2.95.

A reprint of the 1938 edition, a later reprinting (1946) having been reviewed in MR **8**, 365.

12107:

Hadley, H. ★Linear algebra. Addison-Wesley Series in Industrial Management. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1961. ix + 290 pp. $6.75.

This text is written primarily for non-mathematics students interested in learning the basic theory of linear algebra. As a result, the emphasis is on computational procedures and the development of an intuitive appreciation for the subject, rather than upon mathematical elegance and rigor. Illustrative examples appear throughout, many chosen from the area of linear programming.

Discussion of vectors is limited to geometric vectors in E^n, and no distinction is made between vectors in $V_n(R)$ and points in E^n. The notion of basis is defined only for E^n, and linear transformations are discussed as transformations of E^n rather than of vector spaces in general. The treatment of determinants and matrices is the usual one found in elementary texts, with the exceptions that the Laplace expansion of determinants is emphasized, and considerable space is given to methods of inverting non-singular matrices. Simultaneous linear equations are treated thoroughly. Here, in keeping with the policy of the author, practical procedures for solving systems precede the development of necessary and sufficient conditions for the existence of solutions.

An unusual feature of the book is the inclusion, in Chapter 6, of an introduction to the theory of convex sets. This section is considerably more difficult than the rest of the book and represents a deviation from the central theme of linear algebra. For those interested in linear programming, however, this will provide a welcome source for the basic theorems and concepts essential to the subject. The final chapter treats quadratic forms and includes a discussion of eigenvectors for symmetric matrices. *J. E. Whitesitt* (Bozeman, Mont.)

12108:

Černikov, S. N. Contraction of systems of linear inequalities. Dokl. Akad. Nauk SSSR **131** (1960), 518–521 (Russian); translated as Soviet Math. Dokl. **1**, 296–299.

Let $\mathscr{L}$ be a real vector space. Consider a system (1) $f_j(x) \leqq a_j$ $(j=1, 2, \cdots, n)$ of linear inequalities on $\mathscr{L}$, where f_j are linear forms on $\mathscr{L}$ and a_j are real numbers. Let $\mathscr{U}$ be a linear subspace of $\mathscr{L}$ such that not all f_j vanish identically on $\mathscr{U}$. Let s be the rank of the restrictions of $f_1, \cdots, f_n$ on $\mathscr{U}$. The $\mathscr{U}$-contraction of (1) is the system of all inequalities of the form

$$\sum_{k=1}^{s+1} t_k f_{j_k}(x) \leqq \sum_{k=1}^{s+1} t_k a_{j_k},$$

where the restrictions of $f_{j_1}, \cdots, f_{j_{s+1}}$ on $\mathscr{U}$ are of rank s, $t_k \geqq 0$, $\sum_{k=1}^{s+1} t_k > 0$ and $\sum_{k=1}^{s+1} t_k f_{j_k}$ vanishes identically on $\mathscr{U}$. The $\mathscr{U}$-contraction of (1), if it exists, is of rank less than the rank of (1). The given system (1) is studied by means of its contractions. There is a brief discussion of application of successive contractions to linear programming problems. *Ky Fan* (Argonne, Ill.)

12109:

Пародн, М. [Parodi, Maurice]. ★Локализация характеристических чисел матриц и ее применения [La localisation des valeurs caractéristiques des matrices et ses applications]. Translated from the French by P. D. Kalafati and P. A. Švarcman; edited by M. G. Krein. Izdat. Inostr. Lit., Moscow, 1960. 170 pp. 6.40 r.

Translation into Russian of M. Parodi's *La localisation des valeurs caractéristiques des matrices et ses applications* [Gauthier-Villars, Paris, 1959; MR **22** #1587].

12110:

Gyires, Béla. Über die Schranken der Eigenwerte von Hypermatrizen. Publ. Math. Debrecen **7** (1960), 374–381.

Let B be an np-square complex matrix, partitioned into blocks B_{ij}, $i, j=1, \cdots, n$. Each B_{ij} is a p-square matrix and is a product: $B_{ij} = P_{ij} A_{ij} P_{ij}^{-1}$, where $P_{ji}^* = P_{ij}^{-1}$. The author gives bounds on the eigenvalues of B, which depend on the matrices A_{ij} and P_{ij}.

More specifically, let $\lambda(A)$ denote any eigenvalue of the matrix A, let $R(A) = \sup |(Ax, y)|$ and let $M(A) = \sup |(Ax, x)|$, where x, y range over all unit vectors. The main result is: $|\lambda(B)| \leqq M(C)$, where $C = (c_{ij})$ is n-square, $c_{ii} = M(A_{ii})$ and $c_{ij} = R(A_{ij})[\lambda_{\max}(P_{ij}P_{ij}^*)/\lambda_{\min}(P_{ij}P_{ij}^*)]^{1/2}$ for $i \neq j$. Some best possible inequalities are given for the case in which each $A_{ij} = A_{ji}^*$.

B. N. Moyls (Vancouver, B.C.)

12111:

Chalk, J. H. H. A note on a discriminant inequality. Canad. Math. Bull. **3** (1960), 7–9.

Let c_n be the least value of C for which

$$\{\prod_{r \neq s} |z_r - z_s|\}^{1/(n^2-n)} \leqq C\{n^{-1} \sum |z_r|^2\}^{1/2}$$

for all sets of n complex numbers z_r $(r=1, \cdots, n)$. The author considers the conjecture that $c_n = n^{1/(n-1)}$ (suggested by a known result for the restricted case in which $|z_1| = \cdots = |z_n| = 1$). The conjectured equality holds trivially for $n=2$ and the author verifies it for $n=3$, but disproves it for $n \geqq 6$, showing quite simply that $n\{n/(n-1)\}^{n/2-1} \leqq c_n{}^n \leqq \{2n/(n-1)\}^{n/2}$. He also indicates an example showing that $\limsup c_n \geqq 1.05\cdots$, and mentions a result of the reviewer that suggests that $\limsup c_n = (2/\sqrt{e})^{1/2} = 1.10\cdots$. {In a more recent paper [J. London Math. Soc. **35** (1960), 241–250; MR **22**, #4703] the reviewer has shown that

$$(2/\sqrt{e})^{1/2} n^{(1-\eta)/2n} \leqq c_n \leqq (2/\sqrt{e})^{n/2(n-1)} n^{n/(n-1)} (n-1)^{-1}$$

$(n=2, 3, \cdots)$, where $\eta \to 0$ $(n \to \infty)$.}

H. P. Mulholland (Exeter)

12112:

Mirsky, L. Maximum principles in matrix theory. Proc. Glasgow Math. Assoc. **4**, 34–37 (1958).

For a complex matrix A of order n, let $s_1(A) \geqq \cdots \geqq s_n(A)$ denote the singular values of A, and let $[A]_k$ denote the submatrix of order k $(1 \leqq k \leqq n)$ on the upper left corner of A. Write $E_j(z_1, \cdots, z_n)$ for the jth elementary symmetric function of $z_1, \cdots, z_n$, and $E_j(A)$ for the jth elementary symmetric function of the eigenvalues of A. Generalizing a result of the reviewer [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 760–766; MR **13**, 661], M. Marcus and B. N. Moyls [Canad. J. Math. **9** (1957), 313–320; MR **19**, 114] obtained the following theorem. Let $A_1, A_2, \cdots, A_m$ be m complex matrices of order n. Let $\sigma_i = \prod_{r=1}^{m} s_i(A_r)$ and $1 \leqq j \leqq k \leqq n$. Then

$$\max |E_j([U_1 A_1 \cdots U_m A_m U_{m+1}]_k)| = E_j(\sigma_1, \cdots, \sigma_k),$$

where the maximum is taken over all sets of unitary matrices $U_1, U_2, \cdots, U_{m+1}$. A new proof of this theorem is given in the present paper. The author also proves the following result. Let N be a normal matrix of order n with eigenvalues λ_i, which are arranged in non-increasing order of absolute values. Then for $1 \leqq j \leqq k \leqq n$ and for any positive integer r,

$$\max E_j([V^*(N^*U^*)^r(UN)^r V]_k) = E_j(|\lambda_1|^{2r}, \cdots, |\lambda_k|^{2r}),$$

where the maximum is taken over all pairs of unitary matrices U, V. *Ky Fan* (Detroit, Mich.)

12113:

Bullen, Peter; Marcus, Marvin. Symmetric means and matrix inequalities. Proc. Amer. Math. Soc. **12** (1961), 285–290.

Let (a) denote the n-tuple of positive numbers $(a_1, \cdots, a_n)$, $(\bar{a})$ the $(n+1)$-tuple of positive numbers $(a_1, \cdots, a_n, a_{n+1})$, $E_r(a)$ the rth elementary symmetric function of the (a), $p_r(a) = \binom{n}{r}^{-1} E_r(a)$, $F_{r,p} = (E_r/E_{r-p})^{1/p}$; (a) is proportional to (b) if there exists λ such that $a_i = \lambda b_i$, $i=1, 2, \cdots, n$.

The following are proved. (1) If $1 \leqq r \leqq k \leqq n$, then $p_r^k(a)/p_k r(a) \leqq p_r^{k+1}(\bar{a})/p_{k+1}r(\bar{a})$, with equality only if $a_1 = \cdots = a_{n+1}$. (2) If $\mu_1 \geqq \cdots \geqq \mu_{n+1} > 0$ are the eigenvalues of a positive definite Hermitian matrix H and $\lambda_1 \geqq \cdots \geqq \lambda_n > 0$ the eigenvalues of a principal submatrix of H, then $p_1^k(\lambda)/p_k(\lambda) \leqq p_1^{k+1}(\bar{\mu})/p_{k+1}(\bar{\mu})$. (3) If $1 \leqq p \leqq r \leqq n$, then $F_{r,p}(a+b) \geqq F_{r,p}(a) + F_{r,p}(b)$, with equality only if $r=1$, $p=1$ or (a) is proportional to (b). (4) If A and B are positive definite Hermitian matrices with eigenvalues $0 < \alpha_1 \leqq \cdots \leqq \alpha_n$, $0 < \beta_1 \leqq \cdots \leqq \beta_n$, respectively, and $C = A + B$ has eigenvalues $0 < \delta_1 \leqq \cdots \leqq \delta_n$, then for $1 \leqq p \leqq r \leqq k \leqq n$

$$F_{r,p}(\delta_1, \cdots, \delta_k) \geqq F_{r,p}(\alpha_1, \cdots, \alpha_k) + F_{r,p}(\beta_1, \cdots, \beta_k).$$

A. E. Danese (Schenectady, N.Y.)

12114:

Clerc, D. La recherche des valeurs propres dominantes et subdominantes d'une matrice réelle ou complexe. Rech. Aéro. No. 72 (1959), 53–58.

L'auteur étend aux matrices à termes complexes les méthodes des matrices allongées, présentées antérieurement [Rech. Aéro. No. 54 (1956), 39–48; No. 56 (1957), 33–40; MR **18**, 766] pour les matrices à valeurs propres réelles et distinctes. *J. Kuntzmann* (Grenoble)

12115:

Gutnik, L. A. Some questions in the arithmetic of matrices. Moskov. Gos. Ped. Inst. Uč. Zap. **138** (1958), 47–77. (Russian)

12116:

Gutnik, L. A. On some number-theoretic functions of matrices. Moskov. Gos. Ped. Inst. Uč. Zap. **138** (1958), 79–106. (Russian)

12117:

Trupin, Š. D. [Trupins, Š.]. On a generalization of theorems about divisibility of a multilinear function in a linear unbounded space. Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), 13–28. (Russian. Latvian summary)

On trouve les conditions pour la validité de l'équation $\sum_{i=1}^{n} \omega_i{}^m x \cdots x \cdot \alpha_i x = 0$ où $\omega_i{}^m$ est une fonction m-linéaire et $\alpha_i x$ une fonction linéaire scalaire. Voir le travail de l'auteur [Latvijas PSR Zinātņu Akad. Vēstis **1958**, no. 8 (133), 82–92; MR **20** #6711]. *A. Švec* (Prague)

12118:

Tenca, Luigi. Classi di determinanti che hanno come caso particolare quelli di Hankel, da lui detti ortosimmetrici. Period. Mat. (4) **37** (1959), 101–109.

ASSOCIATIVE RINGS AND ALGEBRAS

See also 12129, 12130.

12119:

Zawadowski, W. A theorem on quasi-ordered division algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 173–177. (Russian summary, unbound insert)

The main result of this paper is the theorem that any quasi-ordered, real division algebra with a positive identity which is a strong order unit necessarily consists of all real multiples of the identity. Several corollaries follow from this result. For instance, if R is a field which is a quasi-ordered rational algebra, and if the identity of R is positive and a strong order unit, then R is isomorphic to a subfield of the real numbers. {However, the isomorphism will not in general be order-preserving, as the paper implies. For example, let $F = Q(\sqrt{2})$ with positive cone $P = \{r + s\sqrt{2} : r + s\sqrt{2} \geqq 0 \text{ and } r - s\sqrt{2} \geqq 0\}$. Also, the proofs of Corollaries 1 and 2 seem to require the added hypothesis that R is Archimedian.}

R. S. Pierce (Seattle, Wash.)

12120:

Wall, Drury W. Cartan invariants of algebras with unique minimal faithful representations. Illinois J. Math. **4** (1960), 133–142.

Let $\mathfrak{A}$ be a finite-dimensional algebra with unit element over a field K and with radical $\mathfrak{N}$. Let $\{e_i\}_{i=1}^n$ be a maximal set of non-isomorphic primitive idempotents of $\mathfrak{A}$. Let c_{ik} and $\tilde{c}_{ik}$ be the multiplicity of the irreducible left $\mathfrak{A}$-module $\mathfrak{A}e_k/\mathfrak{N}e_k$ and the irreducible right $\mathfrak{A}$-module $e_k\mathfrak{A}/e_k\mathfrak{N}$ in (the composition factor module series of) $\mathfrak{A}e_i$ and $e_i\mathfrak{A}$ respectively; they are called the left and the right Cartan invariants. (The author's statement in § 1 that $c_{ik} = \tilde{c}_{ki}$ is not true unless, for example, K is assumed to be algebraically closed [cf. Nakayama, Ann. of Math. (2) **39** (1938), 361–369].) Similarly, we denote by g_{ik} and $\tilde{g}_{ik}$ the multiplicity of $\mathfrak{A}e_k/\mathfrak{N}e_k$ and $e_k\mathfrak{A}/e_k\mathfrak{N}$ in the socle of $\mathfrak{A}e_i$ and $e_i\mathfrak{A}$ respectively. Put then $\Sigma_i = \{k | g_{ik} \neq 0\}$ and $\Pi_i = \{k | \tilde{g}_{ik} \neq 0\}$, and if Σ_i or Π_i consists of only one element denote it also by $\sigma(i)$ or $\pi(i)$, respectively. Suppose $\mathfrak{A}e_i$ is dominant, i.e., $\mathfrak{A}e_i$ is dual to a primitive right ideal, say $e_j\mathfrak{A}$. Then $c_{ik} = \tilde{c}_{jk}$ for all k. Moreover, $\mathfrak{A}e_i$ and $e_j\mathfrak{A}$ have a unique minimal left and a unique minimal right subideal, which are isomorphic to $\mathfrak{A}e_k/\mathfrak{N}e_k$ and $e_k\mathfrak{A}/e_k\mathfrak{N}$, respectively, and thus $\sigma(i) = j$ and $\pi(j) = i$. Now, a primitive left ideal $\mathfrak{A}e_i$ is weakly subordinate to the set of dominant left ideals [Wall, Duke Math. J. **25** (1958), 321–329; MR **20** #3190] if and only if $e_k\mathfrak{A}$ is dominant for all $k \in \Sigma_i$, and indeed, in this case, $\mathfrak{A}e_i$ is embeddable into the direct sum $\sum_{k \in \Sigma_i} \bigoplus (\mathfrak{A}e_{\pi(k)})^{g_{ik}}$, which implies the inequality $c_{ij} \leqq \sum_{k=1}^n g_{ik}\tilde{c}_{kj}$ for all $j = 1, 2, \cdots, n$, with equality holding for all j if and only if $\mathfrak{A}e_i$ is dominant.

G. Azumaya (Sapporo)

12121:

Samuel, Pierre. On unique factorization domains. Illinois J. Math. **5** (1961), 1–17.

The main purpose of the author is to answer the following question proposed by W. Krull [Math. Z. **43** (1938), 768–782]: Is the power series ring over a UFD (unique factorization domain) also a UFD? The author uses a reduction process. In the first step he proves the following reduction theorem: (A) If A is a Macaulay UFD such that $A_{\mathfrak{p}}[[X]]$ is a UFD for every prime ideal $\mathfrak{p}$ of height 2, then $A[[X]]$ is a UFD. (A) reduces the above question to the case in which A has dimension 2. Secondly, he proves the following main theorem: (B) Let A be a domain, a, b, c three elements of A, and i, j, k three integers such that a is prime, b and c are relatively

prime, $a^{i-1} \notin Ab + Ac$, $a^i \in Ab^k + Ac^j$ and (*) $ijk - ij - jk - ki \geq 0$. Then $A[[X]]$ is not a UFD. Now, in order to disprove the above conjecture it is sufficient to construct a UFD containing three elements satisfying the assumptions of (B). The following theorem (C) is proved: If k is a perfect field of characteristic two, then the ring $B = k[x, y, z]$, where $z^2 - x^3 - y^7 = 0$, is a UFD, and so is the local ring $B_{(x,y,z)}$. The author makes the following interesting conjecture. It may be possible to attach to every UFD A a numerical invariant I (generalizing (*)) such that $A[[X]]$ is a UFD for $I < 0$ and is not for $I \geq 0$. He proposes also the following open questions. (1) Let A be a local domain of dimension ≥ 4 which is a factor ring $R/(f)$ of a regular local ring by a principal ideal. If $A_\mathfrak{p}$ is regular for every non-maximal prime ideal $\mathfrak{p}$, will A be a UFD? (2) Are all the UFD Macaulay rings?

The author gives also two simple counterexamples to show that unique factorization is preserved neither by ground-field extension nor by ground-field restriction.

P. Abellanas (Madrid)

12122:

Thimm, Walter. Teoria degli anelli di serie di potenze. Ann. Mat. Pura Appl. (4) **52** (1960), 247–330.

Les §§ 1–8 contiennent un exposé des propriétés classiques des anneaux de séries formelles: substitutions, jacobiens, caractère noethérien, lemme de Hensel, théorème de préparation de Weierstrass, factorialité des anneaux de séries formelles sur un corps infini (l'hypothèse que le corps est infini est inutile), le fait que l'extension aux séries formelles d'un idéal premier d'un anneau de polynômes est un idéal quasi-premier. Les §§ 9–11 donnent la structure des idéaux premiers d'un anneau de séries formelles, et l'appliquent aux variétés algébroïdes. Au § 12, s'inspirant de la construction des formes associées de Chow-Van der Waerden, l'auteur définit, pour tout idéal premier P de $K[[X_1, \cdots, X_n]]$, un polynôme homogène qu'il appelle sa "fonction norme". Le § 13 traite de la dimension des idéaux et des dimensions d'intersections, et la § 14 des nappes algébroïdes d'une variété algébrique.

Un second chapitre s'occupe des anneaux de séries entières convergentes; le théorème de Weierstrass montre qu'ils sont noethériens, et le lemme de Hensel n'en fait pas sortir. Etude des germes d'ensembles analytiques au moyen de la fonction norme. Le complémentaire d'une sous-variété analytique (sur **C**) est connexe par arcs. L'ensemble des points simples d'une variété analytique irréductible est connexe par arcs.

P. Samuel (Clermont-Ferrand)

12123:

Faith, Carl C. Abelian Galois groups. Proc. Amer. Math. Soc. **10** (1959), 767–774.

Soient K une algèbre unitaire et centrale sur un corps C, F un sous-anneau galoisien de K tel que le groupe de Galois $G = G(K/F)$ de K/F soit abélien, A le centralisateur de F dans K, R le radical de A. L'algèbre K/C est supposée, en général, satisfaire à la condition minimale, bien que pour certains résultats intermédiaires on ne l'impose qu'à la C-algèbre $A = A/R$.

Le résultat principal du travail est que si K/C est une algèbre simple de rang fini, la commutativité de G entraîne, sauf dans un cas très particulier (à savoir, quand C est le corps GF(4) de 4 éléments et A est semi-simple), que ce groupe est complètement externe (c'est-à-dire, ne comporte aucun automorphisme intérieur de K autre que l'identité) ou complètement interne (c'est-à-dire, est un groupe de tels automorphismes). Idée de démonstration: X étant un sous-anneau unitaire de K, X^* étant le groupe des $x \in X$ qui sont les unités de K, et σ étant un automorphisme de K commutant avec tous les automorphismes intérieurs I_x produits par les $x \in X^*$, on démontre que, si le radical de X n'est pas 0, σ préserve tout $x \in X^*$ et que, dans le cas contraire, ceci a encore lieu si $x - 1 \in X^*$. En particulier, si X est engendré par les $x \in X^*$ (ce qui a lieu si $C \neq \mathrm{GF}(2)$; pour ce dernier cas, l'auteur donne un contre-exemple) et a un radical non-nul, ou s'il est engendré par de tels x satisfaisant à $x - 1 \in X^*$ (ce qui a lieu si C a plus de 4 éléments et X est semi-simple), un tel σ préserve tout $x \in X$. On se trouve dans la situation précédente si, sous l'hypothèse que G n'est pas complètement externe (ce qui exclue $A \subseteq C$), on pose $X = A$ et $\sigma \in G$, car alors I_x, $x \in A^*$, est $\in G$, lequel est abélien. Par suite, si C a plus de 4 éléments et si A satisfait à la condition minimale (auquel cas il est semi-simple si $R = 0$), tout $\sigma \in G$ préserve tout $x \in A$, d'où résulte $A \subseteq F$ et, a fortiori, $C \subseteq F$. Ce dernier résultat est encore vrai si C est premier, en particulier si $C = \mathrm{GF}(2)$ ou GF(3). Ainsi, si $C \neq \mathrm{GF}(4)$, on a $G \subseteq G(K/C)$. Il n'y a plus qu'à appliquer le théorème bien connu que si K/C est simple et de rang fini, $G(K/C)$ est complètement interne. L'auteur donne un contre-exemple (exact, mais dont la démonstration comporte une petite lacune) dans le cas exceptionnel $C = \mathrm{GF}(4)$.

En se servant des résultats de Köthe (qui a montré qu'il existe des algèbres simples K/C de rang infini telles que $G(K/C)$ ne soit pas complètement interne et possède des sous-groupes H complètement externes), l'auteur donne (en se basant sur un critère de Nakayama), dans le cas où il existe de tels H d'ordre fini, des conditions suffisantes pour qu'il existe des sous-anneaux galoisiens F de K tels que $G(K/F)$ ne soit complètement interne ni complètement externe, et il montre qu'il existe aussi bien des algèbres de division que des algèbres simples qui ne le sont pas, satisfaisant à ces conditions.

M. Krasner (Paris)

12124:

Bortfeld, Reinhard. Ein Satz zur Galoistheorie in Schiefkörpern. J. Reine Angew. Math. **201** (1959), 196–206.

Soit K un corps (qu'on ne suppose pas commutatif) et soit ω un automorphisme de K tel que K soit de rang fini (par exemple à gauche) par rapport à son corps des invariants L. L'auteur démontre, sous l'hypothèse que le centre C de K soit un corps infini, que tout corps Z entre L et K est le corps des invariants d'un automorphisme convenable de K (appartenant manifestement au groupe de Galois $G(K/L)$ de K/L), ce qui généralise le résultat du cas commutatif qu'une extension cyclique l'est encore par rapport à tout corps intermédiaire. Dans le cas où C est un champs de Galois, l'auteur énonce sans démonstration les conditions nécessaires et suffisantes pour que le même résultat ait lieu.

L'idée de démonstration: Soit $J(G)$ le groupe des automorphismes intérieurs $\in G = G(K/L)$. Alors, en vertu de la théorie de H. Cartan, ω est d'ordre fini, soit a, par rapport à $J(G)$, et si $\omega^a = \sigma_\lambda$ est l'automorphisme $\in J(G)$ engendré par un $\lambda \in K$ convenable, on voit facilement que λ est algébrique sur C et que le centralisateur L' de L dans K

(d'une manière générale, on va noter Z' le centralisateur dans K de tout corps Z intermédiaire entre L et K) est le corps commutatif $C(\lambda)$. $\bar{\omega}$ étant l'automorphisme induit par ω (par rapport auquel L' est stable) dans L', $L \cap L'$ est le corps des invariants de $\bar{\omega}$ dans L', donc un sous-corps galoisien de L', et $G(L'/L \cap L')$ est engendré par $\bar{\omega}$. Si b est l'ordre de $\bar{\omega}$, on a $b|a$, car $\omega^a = \sigma_\lambda$ et $\lambda \in L'$. Par suite, $L'/L \cap L'$ est séparable et de degré $b < +\infty$, et il résulte de ce fait et de $L \cap L' = (C \cap L)(\lambda)$ que L', et, a fortiori, C est une extension algébrique simple de $C \cap L$, $C/C \cap L$ étant, en particulier, une extension cyclique de degré b, dont le groupe est canoniquement isomorphe à $G(L'/L \cap L')$. Les corps intermédiaires de $C/C \cap L$ sont donc les corps C_ν des invariants de $\bar{\omega}^\nu$ dans C, où $\nu|b$. Si $L \subseteq Z \subseteq K$, Z sera dit de la classe ν si $C \cap Z \supseteq C \cap L$ est C_ν. Comme tous les éléments de $G(K/Z) \subseteq G(K/L)$ sont de la forme $\sigma_\alpha \omega^t$ ($\alpha \in L'$, $0 \leq t < a$), on montre facilement que si d est le plus grand commun diviseur des t tels que $\sigma_\alpha \omega^t \in G(K/Z)$, ce d est parmi ces t et coïncide avec la classe ν de Z. Le corps des invariants de $\omega' = \sigma_\alpha \omega^\nu$ correspondants est $\supseteq Z$ et a la même classe ν. Par suite, si Z est un corps maximal de sa classe, il doit être le corps des invariants d'un tel ω'.

Supposant Z de classe 1, si $M \supseteq Z$ est un corps maximal de cette même classe et si $\omega' = \sigma_\alpha \omega$ est un automorphisme dont M est le corps des invariants, le reste de la démonstration (pour $\nu = 1$) consiste à construire un $\beta \in Z'$ n'appartenant au centralisateur d'aucun corps entre M (inclu) et Z (exclu) [le nombre de ces corps est fini, car on a $L' \supseteq Z' \supseteq M' \supseteq C$ et L'/C est une extension commutative de degré fini], auquel cas le corps des invariants de $\sigma_{\beta\alpha}\omega = \sigma_\beta \omega^\nu$ se réduit visiblement à Z. Cette construction, techniquement compliquée, mais de type bien connue, présuppose que C soit infini. Le cas $\nu \neq 1$ se réduit au cas $\nu = 1$ en remplaçant ω par ω^ν.

{Erratum remarqué: à la p. 206, ligne 6, il faut lire: $\beta \to \beta^{\frac{\alpha\alpha'}{b}}$, au lieu de: $\beta \to \beta^{\frac{\alpha\alpha'}{b}}$.} *M. Krasner* (Paris)

12125:

Takasaki, Tatsumi. Some properties of the ring with semi-valuation. Bull. Fukuoka Gakugei Univ. III **9** (1959), 5–13. (Japanese)

A semi-valuation on a ring A is a non-negative real-valued function w on A such that $w(0)=0$, $w(x-y) \leq w(x)+w(y)$ and $w(xy) \leq w(x) \cdot w(y)$ for all x and y in A. Given a semi-valuation w, one can define a pseudo-metric $(x, y) \to w(x-y)$ on A, so that A is a topological ring under the resulting topology. The present paper gives some theorems concerning semi-valuation rings which are generalizations of theorems on normed rings. The proofs are mostly straightforward, and no application is given.

I. Namioka (Ithaca, N.Y.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 12050.

12126a:

Hsiang, Wu-Yi. On the distributive law. Proc. Amer. Math. Soc. **11** (1960), 348–355.

12126b:

Hsiang, Wuchung; Hsiang, Wuyi. A note on the theory of (m, n)-distributive rings. Arch. Math. **11** (1960), 88–90.

These papers study the distributive laws by looking at (m, n)-rings $(R, +, \cdot)$, where $(R, +)$ is an abelian group, and $\cdot$ is an operation on R satisfying the condition

$$(D_{m,n}) \qquad \Big(\sum_{i=1}^{m} x_i\Big)\cdot\Big(\sum_{j=1}^{n} y_j\Big) = \sum_{i=1,j=1}^{m,n} x_i \cdot y_j,$$

for x_i, y_j in R. In the first paper, the author observes that there is a relation between (m, n)-rings $(R, +, \cdot)$ and rings $(R, +, *)$; in particular, for every one of the first kind there is one of the second kind, having the same underlying group structure, such that $a*b = a \cdot b - a \cdot 0 - 0 \cdot b + 0 \cdot 0$ for all a, b in R. He also gives conditions under which all systems $(R, +, \cdot)$ satisfying one sequence for $D_{m,n}$-laws can satisfy another such sequence. In the second paper the study of (m, n)-rings is carried further.

D. W. Jonah (Providence, R.I.)

HOMOLOGICAL ALGEBRA

12127:

★Séminaire A. Grothendieck; 1re année: 1957. Algèbre homologique. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. 42 pp. (mimeographed)

Two lectures by A. Andreotti: "Généralités sur les catégories abéliennes"; and one by P. Cartier: "Les groupes $\mathrm{Ext}^p(A, B)$".

12128:

Papp, Zoltán. On algebraically closed modules. Publ. Math. Debrecen **6** (1959), 311–327.

This paper considers the structure of injective modules, and, in particular, their decomposition (when possible) into direct sums of indecomposable injective modules. Sections 1 and 2 are preliminary. The theorem of section 3 asserts that a ring R is Noetherian if and only if any direct sum of injective R-modules is again an injective R-module. This result was credited to H. Bass by S. U. Chase, Trans. Amer. Math. Soc. **97** (1960), 457–473 [MR **22** #11017] where a proof also appears. However, the present author appears to have priority. Unfortunately, the author was unaware until too late that most of the results of his section 4 had already appeared in a paper by the reviewer [Pacific J. Math. **8** (1958), 511–528; MR **20** #5800]. *E. Matlis* (Evanston, Ill.)

12129:

Rosenberg, Alex; Zelinsky, Daniel. On Amitsur's complex. Trans. Amer. Math. Soc. **97** (1960), 327–356.

Let F be a commutative C-algebra possessing the same unit. The reviewer has introduced in same Trans. **90** (1959), 73–112 [MR **21** #78], denoted by A., the following complex $\mathcal{C}(F/C)$: $1 \to C^* \to F^* \to \cdots \to (F^n)^* \to \cdots$, where $(F^n)^*$ is the multiplicative group of all invertible elements of the ring $F^n = F \otimes_C \cdots \otimes_C F$, $F^0 = C$. The derivation Δ_n: $(F^n)^* \to (F^{n+1})^*$ is given by $\Delta_n(x) = [\varepsilon_1(x)\varepsilon_3(x)\cdots] \times [\varepsilon_2(x)\varepsilon_4(x)\cdots]^{-1}$, where ε_i: $F^n \to F^{n+1}$ is the homo-

morphism obtained by inserting 1 in the ith place, $i=1, 2, \cdots, n+1$. The cohomology groups of this complex will be denoted by $H^n(F/C)$. In the present paper the authors extend and simplify the results and proofs of A., and even set on the right track one erroneous result of the reviewer. The significance of the complex $\mathcal{C}(F/C)$ is that the cohomology group $H^2(F/C)$ is isomorphic with the Brauer group of all central simple C-algebras split by F, if F is a field. This is extended as follows: A C-algebra is split if it is isomorphic with $\mathrm{Hom}_C(V, V)=\mathrm{End}_C(V)$ for some finitely generated faithful projective C-module V, and a C-algebra A is said to be split by F if $A \otimes_C F$ is a split F-algebra. The full Brauer group $\mathcal{B}(F/C)$ of all central separable C-algebras split by F is the group of all these algebras with respect to the tensor product and to the equivalence relation: $A \sim B$ if and only if $A \otimes E \cong B \otimes E'$ for some split C-algebras. Now if V is a C-module, there are two natural homomorphisms η_2, η_3 of $\mathrm{End}_F(V \otimes F)$ into $\mathrm{End}(V \otimes F^2)$ induced by the two maps of $V \otimes F$ into $V \otimes F^2$. If $V=F$ and $t \in F^3$, let $L(t) \in \mathrm{End}_{1 \otimes F^2}(F^3)$ be the multiplication in F^3 by t_1. The first result of the authors is that if F is a finitely generated projective C-algebra such that the unit map $C \to F$ splits, then the correspondence $t \to A(t)=\{a \in \mathrm{End}_{1 \otimes F}(F^2) \mid L(t)(\eta_2 a)=(\eta_3 a)L(t)\}$, $t \in F^3$, induces a homomorphism α: $H^2(F/C) \to B(F/C)$. Furthermore, if C, F, $F \otimes F$ are direct sums of rings over which every finitely generated faithful projective module is free, then α is actually an isomorphism. For fields F this was shown in A. This general class of C-algebras includes, e.g., polynomial rings $K[x]$, where K is a commutative ring with the minimum condition, and semi-local rings C.

Next, the author shows that if F is a separable finite field extension of C which can be embedded in a normal extension K whose Galois group is G, and $H \subseteq G$ is the group leaving the elements of F invariant, then $H^n(F/C) \cong H^n([G:H], K^*)$, where the latter are the Adamson cohomology groups of F [Proc. Glasgow Math. Assoc. **2** (1954), 66–76; MR **16**, 442]. The case $F=K$ was covered in A., but the present proof is simpler. Now, let F be an inseparable field extension of C of exponent 1. Hochschild [Trans. Amer. Math. Soc. **79** (1955), 477–489; MR **17**, 61] has given a representation of the Brauer group $\mathcal{B}(F/C)$ by the group $\mathcal{E}(T, F)$ of all regular restricted Lie algebra extensions of F by the set of all derivations T of F mapping C onto the zero. A straight identification of $H^2(F/C)$ and $\mathcal{E}(T, F)$ has been tried unsuccessfully and in a wrong way in A. This has been accomplished explicitly in the present paper. Furthermore, the authors have obtained an explicit form for representatives of the cocycles in this case, which as a by-result yields the following classical result: If a central simple algebra A is split by an inseparable field extension $F=C[a_1, \cdots, a_n]$, $a_i \in C$, then A is isomorphic with $A_1 \otimes \cdots \otimes A_n$, where A_i is a central simple C-algebra split by the field $C[a_i]$. Finally, they apply the complex $\mathcal{C}(F/C)$ to obtain an alternative proof to the fact that the natural homomorphism of the full Brauer group $\mathcal{B}(C[x])$ over the polynomial ring $C[x]$ (C – a field) into $\mathcal{B}(C)$ is a monomorphism if and only if C is a perfect field.

S. A. Amitsur (Jerusalem)

12130:

Auslander, Maurice; Goldman, Oscar. The Brauer group of a commutative ring. Trans. Amer. Math. Soc. **97** (1960), 367–409.

General theory of separable algebras over commutative rings is given; an algebra Λ over a commutative ring R is called separable if Λ is projective over the enveloping algebra $\Lambda^e=\Lambda \otimes_R \Lambda^0$, Λ^0 being the opposite of Λ. It is shown that Λ is separable over R if and only if Λ is so over its center C, and C is so over R, and thus the theory can substantially be split into two parts, i.e., commutative and central cases. The former has recently been studied to some extent by Auslander and Buchsbaum [Amer. J. Math. **81** (1959), 749–765; MR **21** #5659], while the latter forms a generalization of the classical theory of central-simple algebras (over a field) and their Brauer group. First, besides formal properties of separable algebras concerning epimorphism, tensor product and homological dimension, etc., several criteria for separability are given. For instance, Λ is separable over its center C if and only if $\Lambda \otimes_C \Lambda^0$ is canonically isomorphic with $\mathrm{Hom}_C(\Lambda, \Lambda)$ and Λ is C-projective; this shows that "central-separable" (c.-s.) includes the "maximally central" of Azumaya [Nagoya Math. J. **2** (1951), 119–150; MR **12**, 669], whose results and methods, it is stated, could be used to derive some results in the present paper. Every endomorphism of a c.-s. algebra is an automorphism, and it is indeed inner under a certain condition on projective modules over the ground ring, i.e., the center. Commutants in a c.-s. algebra behave similarly to the classical case of those in a central-simple algebra. Two-sided ideals in a c.-s. algebra are in 1-1 correspondence with those in the center. For any algebra Λ over R, a certain ideal $\mathfrak{d}(\Lambda/R)$ in the center C is defined and is called the homological different; $\mathfrak{d}(\Lambda/R)$ coincides with C if and only if Λ is separable over R. It is used for studying "ramification", and Λ is thus shown to be separable if and only if, for every maximal ideal $\mathfrak{m}$ of R, $\Lambda \otimes_R R_{\mathfrak{m}}$ [resp. $\Lambda/\mathfrak{m}\Lambda$] is so over $R_{\mathfrak{m}}$ [resp. $R/\mathfrak{m}$], provided that R is noetherian and Λ is R-finite. The Brauer group $B(R)$ over R is introduced by defining two c.-s. algebras Λ_1, Λ_2 to be equivalent when $\Lambda_1 \otimes_R \Omega_1 \approx \Lambda_2 \otimes_R \Omega_2$, each of Ω_1, Ω_2 being of the form $\mathrm{Hom}_R(E, E)$ with E finite and projective over R. Classical features of Brauer groups and splitting rings are given. Then the paper turns to the case of a local ring R, and shows, among other things, the existence of a separable splitting ring for a c.-s. algebra over R. Also the homomorphism $B(R) \to B(R/\mathfrak{m})$ is studied and shown to be an isomorphism in case R is complete, $\mathfrak{m}$ being the maximal ideal of R; for this, cf. Azumaya [loc. cit.]. Further, separable algebras over an integrally closed noetherian ring are studied and connected with those over its quotient field. The Brauer group $B(K[x])$ over the polynomial ring $K[x]$ over a perfect field K is proved to be isomorphic to $B(K)$; for a different proof, see Rosenberg and Zelinsky [#12129]. In the special case of a valuation ring R the relationship of hereditary and maximal orders in a c.-s. algebra over R is studied, among others. The appendix develops a Galois theory and closes by establishing an exact sequence $1 \to H^2(G, U(S)) \to B(R) \to B(S)$ for a "Galois" extension S of R having G as Galois group, having $U(S)$ as its group of units and satisfying a certain condition concerning projective modules. In all this the authors' previous paper [Trans. Amer. Math. Soc. **97** (1960), 1–24; MR **22** #8034] is used extensively.

T. Nakayama (Nagoya)

12131:

Thomas, Emery. A note on certain polynomial algebras. Proc. Amer. Math. Soc. **11** (1960), 410–414.

For r a positive integer or ∞ and k an integral domain of characteristic 2, define

$$A_r = k[x_1, \cdots, x_i, \cdots; y_1, \cdots, y_i, \cdots],$$

where $1 \leq i \leq r$ if r is an integer and $1 \leq i < \infty$ if $r = \infty$. Define a derivation β on A_r by setting $\beta x_i = y_i$, $\beta y_i = 0$, $\beta | k = 0$. Then, as a k-module, A_r is the direct sum of the kernel of β and a submodule S of A_r which can be described as follows: Let P be the subalgebra of A_r spanned by monomials in the x_i^2. S is the submodule of A_r spanned by elements $px_{i_1} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$, where $p \in P$, $i_1 < \cdots < i_a$, $a \geq 1$; $j_1 \leq \cdots \leq j_b$, $b \geq 0$; $i_1 \leq j_1$ if $b > 0$. Also the kernel of β is $P \oplus \beta S$. These algebras occur in algebraic topology as the mod 2 cohomology algebras of the classifying spaces for real oriented vector bundles, and also in Thom's cobordism theory. *N. Stein* (New York)

12132:

Wall, C. T. C. On a result in polynomial rings. Proc. Cambridge Philos. Soc. **56** (1960), 104–108.

Motivated by his work on cobordism, the author studies the following derivation ∂ in the polynomial ring: $\mathscr{P} = \mathbf{Z}[u_1, u_2, u_3, \cdots]$: $\partial(u_1) = 1$, $\partial(u_{i+1}) = u_i$ for $i > 1$. He shows: Image$(\partial) = \mathscr{P}$; there exist homogeneous (with respect to the grading $\dim(u_j) = j$) elements $u_i' \in \ker(\partial)$, $i > 1$, such that the coefficient of u_i in u_i' is 1 if i is not a power of a prime, and is p if $i = p^k$, p prime. For any choice of $\{u_i'\}$ one has $\ker(\partial) = \mathbf{Z}[u_2', u_3', \cdots]$. Also: If $\bar{\partial}$ is the corresponding derivation in the exterior algebra over $\mathbf{Z}_2$ generated by $\bar{u}_i$ $(i = 1, 2, \cdots)$, then $\ker(\bar{\partial})$ is the exterior algebra on $\{\bar{u}_i'\}$, $i \neq 2^k$. The author uses these results to construct a multiplicative spectral sequence with the same E_2 and E_∞ terms as the universal spectral sequence of the symplectic (or unitary) group. It is also compatible with Bott's divisibility theorems, but how much further the similarity goes does not seem to be known.

A. Dold (New York)

GROUPS AND GENERALIZATIONS

See also 12021, 12046, 12058, 12165.

12133:

Novikov, P. S. The unsolvability of the problem of the equivalence of words in a group and several other problems in algebra. Czechoslovak Math. J. **6** (81) (1956), 450–454. (Russian. English summary)

From the author's summary: "The author proved that the problem of the equivalence of words for groups is unsolvable [Trudy Mat. Inst. Steklov. **44** (1955); MR **17**, 706]. A trivial consequence is the unsolvability of the problem of conjugate words in groups. The latter result can, however, be proved directly and much more simply than in the quoted paper. The possibility of a simpler proof of the original problem, i.e., the unsolvability of the problem of the equivalence of words then follows (but in the weaker form which asserts only the non-existence of a universal algorithm permitting decision on equivalence of words for each group with a finite number of generators and defining relations). The present paper is devoted to the main ideas of these proofs.

"The new proof of the unsolvability of the problem of conjugate words is by reducing it to the unsolvability of equivalence of words for Post's systems of production."

12134:

Faermark, D. S. An algorithm for establishing the identity of words in a nilpotent product of groups each given by a finite number of generators and defining relations. Dokl. Akad. Nauk SSSR **137** (1961), 291–294 (Russian); translated as Soviet Math. Dokl. **2**, 282–285.

Let G_1 and G_2 be groups given by a finite number of generators and a finite number of defining relations. Assume that an algorithm for identifying words exists for each of these groups. Let $G = G_1(n)G_2$ be the nth nilpotent product of G_1 and G_2 [Golovin, Mat. Sb. (N.S.) **27** (**69**) (1950), 427–454; Amer. Math. Soc. Transl. (2) **2** (1956), 89–115; MR **12**, 672; **17**, 824]. An algorithm is described for identifying words in G. Since each $g \in G$ has a unique representation of the form $g = g_1 g_2 u$, $g_i \in G_i$, $u \in [G_1, G_2]^G$, the problem is to find an algorithm for $[G_1, G_2]^G$. The author does this, using the fact that $[G_1, G_2]^G$ is nilpotent and some results of Mal'cev [Mat. Sb. (N.S.) **37** (**79**) (1955), 567–572; MR **17**, 345; see also R. C. Lyndon, Proc. Amer. Math. Soc. **3** (1952), 579–583; MR **14**, 242]. Results of Gol'dina [Uspehi Mat. Nauk **13** (1958), no. 3 (81), 183–189; MR **20** #6455] are also used.

R. R. Struik (Vancouver, B.C.)

12135:

Benado, Mihail. Über die allgemeine Theorie der regulären Produkte von Herrn O. N. Golowin. III. Math. Nachr. **21** (1960), 1–36.

The parts of this same title which have appeared are I and II [same Nachr. **14** (1955), 213–234; **16** (1957), 137–194; MR **18**, 871; **20** #1642] and V (in French [Publ. Sci. Univ. Alger. Sér. A **4** (1957), 111–143; MR **20** #6470]. Let G be a group with operator subgroup H. Let $G \circ^h H = G \circ (G \circ^{h-1} H)$ where $G \circ^1 H = G \circ H$, the subgroup of G generated by all commutators $g^{-1}h^{-1}gh$, where $g \in G$, $h \in H$. Call H h-step normal in G if $G \circ^h H \leq H$, so that 1-step normality is the same as ordinary normality. The h-step normal operator subgroups of G form a complete sublattice of the lattice of operator subgroups of G. Let the G_i be a set of operator subgroups of a group G, their group-theoretic union. Let K_1 be the normal subgroup of G which is the normal closure of the group-theoretic union of the various $G_i \circ G_j$, $i \neq j$. Then define K_n recursively by $K_n = G \circ K_{n-1}$ for $n \geq 2$. Let the G^n be given as usual by $G^0 = G$, $G^{n+1} = G \circ G^n$. Then the author shows that $K_r \circ G^s \leq K_{r+s}$ and that $\{G_i, K_h\}/K_h$ is h-step normal in G/K_h. If, for subgroups G_1 and G_2, $G = \{G_1, G_2\}$, then G_i is h-step normal in G if and only if $K_h \leq G_i$. If G is a regular product of G_1 and G_2 in the sense of Golovin [Mat. Sb. (N.S.) **27** (**69**) (1950), 427–454; Amer. Math. Soc. Transl. (2) **2** (1956), 89–115; MR **12**, 672; **17**, 824], then $K_h = \{G_1, G \circ^h G_1\} \cap \{G_2, G \circ^h G_2\}$. Further, G_i is h-step normal in G if and only if $K_h = \{G_i, G \circ^h G_i\}$. In general, if G is a regular product of subgroups G_i, one says that this regular product is an h-step direct product if each $G_i = \{G_i, G \circ^h G_i\}$, so that the 1-step direct products are just the direct products. A necessary and sufficient condition for a regular product to be h-step direct is that K_h be trivial. The $(h-1)$-nilpotent products of Golovin [loc. cit.] are universal as h-step direct products in that each h-step direct product is a

homomorphic image of one of these nilpotent products. Further, an h-step direct product of groups G_i, each of which is an h-step direct product of groups H_{ij}, is an h-step product of the H_{ij}. If G is an h-step direct product of the groups G_i and if H_i is a subgroup of G_i, then $\{H_i\}$ is an h-step direct product of the H_i. A theorem of Baer and Levi [Kurosh, *Theory of groups, II*, Chelsea, New York, 1956; MR **18**, 188] is extended to show that a group which is a free product cannot be reconstituted as an h-step direct product. Suppose that G is the union of subgroups G_i and that K_h is trivial for some index h. Then, if Z_h is the customary member of the ascending central series for G, G/Z_h is the direct sum of all the $\{G_i, Z_h\}/Z_h$. If G is the h-step direct product of these G_i, then Z_h is the group-theoretic union of K_1 and of an h-step direct product of certain endomorphic images of Z_h. The author employs lattice-theoretic concepts as in his paper C. R. Acad. Sci. Paris **244** (1957), 1702–1704 [MR **19**, 528] to obtain some of the results.

F. Haimo (St. Louis, Mo.)

12136:

Hilton, P. J. Remark on free products of groups. Trans. Amer. Math. Soc. **96** (1960), 478–487.

Let $G_1, G_2, \cdots, G_n$ be groups. If $I=\{i_1, i_2, \cdots, i_r\}$ is a set of integers such that $1 \leqq i_1 < i_2 < \cdots < i_r \leqq n$, then let $G_I = G_{i_1} * \cdots * G_{i_r}$. If $J \subset I$, we have the obvious projection $\pi_J{}^I : G_I \to G_J$. Let $[G_I] = \bigcap_{J \subset I} \ker \pi_J{}^I$. The author proves that $*G_i$ is isomorphic as a set (not as a group) to $\times_I[G_I]$. This isomorphism has certain nice properties such as naturality, which enable the author's theorems to be used in the theory of group complexes.

{There are two misprints. On page 478, just before (2.1), "as" should read "is". On page 484, line 9, "λ^q" should read "λ^{q+1}".} *D. B. A. Epstein* (Princeton, N.J.)

12137:

Kappe, Wolfgang. Die A-Norm einer Gruppe. Illinois J. Math. **5** (1961), 187–197.

The norm $N(G)$ of a group G is the intersection of the normalizers of all its subgroups. If E-subgroups of G are those enjoying a certain property E, such as the property A of being abelian, and if E-Sylow subgroups are those E-subgroups not contained in larger E-subgroups, then the E-norm of G is the intersection of the normalizers of its E-subgroups. The A-norm $N_A(G)$ is a characteristic subgroup whose structure and embedding in G are described in several theorems. The only possible A-norms are L-groups, namely, groups in which each element commutes with all its conjugates. The product $N \cdot U$ is an L-group if U is an abelian subgroup of G, and N a normal subgroup contained in $N_A(G)$. Also, for $x \in N$ and $g, h \in G$, the commutator $[x, g] = x^{-1}g^{-1}xg$ commutes with h if $[g, h] = 1$. The element x is called a (right) Engel element (of length 2) if $[[x, g], g] = 1$ for all $g \in G$. Each Engel element x commutes with all its conjugates. Together these conjugates generate an abelian group $\langle x \rangle$ on which the elements of G operate as elements of the endomorphism ring. Thus if $x^g = g^{-1}xg$, then $x^{g^{-1}} = x^{-1}x^g = [x, g]$ and $x^{(g-1)(g^{-1})} = [[x, g], g] = 1 = x^0$, so we may write $(g-1)^2 = 0$ or $g + g^{-1} = 2$. Thus $2 = hg^{-1} + gh^{-1} = h(2-g) + g(2-h)$, so $(g-1)(h-1) = -(h-1)(g-1)$ and $[[x, g], h] = [[x, h], g]^{-1}$. Similarly, $[g, h] - 1 = 2(g-1)(h-1)$, so $[x, [g, h]] = [[x, g], h]^2$. Likewise, $[[x, g], [h, g]] = 1$. Other relations include $[[f, g], h]^3 = 1$ for any triple of elements from an L-group. Three stated theorems elaborate on these results. *J. S. Frame* (Washington, D.C.)

12138:

Diciman, A. P. On p-subgroups and theorems analogous to the Sylow theorem. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 99–114. (Russian)

12139:

Mačul'skiĭ, A. A. On subgroups of a non-solvable group. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 129–133. (Russian)

12140:

Baer, Reinhold. Die Existenz Hallscher Normalteiler. Arch. Math. **11** (1960), 77–87.

This paper is concerned with conditions on a finite group G which, together with $\mathfrak{r}'$-homogeneity, are equivalent to $\mathfrak{r}$-closure. Here $\mathfrak{r}$ is a set of primes that divide the order $o(G)$ of G, and $\mathfrak{r}'$ the complementary set, $\mathfrak{r}$-elements (or $\mathfrak{r}$-subgroups) are those whose orders are divisible only by primes of $\mathfrak{r}$, $n_\mathfrak{r}$ is the greatest $\mathfrak{r}$-factor of n, $G_\mathfrak{r}$ is the set of $\mathfrak{r}$-elements of G, and G is called $\mathfrak{r}$-closed if $G_\mathfrak{r}$ is a subgroup of G. Furthermore, G is called $\mathfrak{r}'$-homogeneous if the normalizer $\mathfrak{N}S$ and centralizer $\mathfrak{C}S$ of any $\mathfrak{r}'$-subgroup S of G are such that $\mathfrak{N}S/\mathfrak{C}S$ is also an $\mathfrak{r}'$-subgroup. To any complete ordering $\mathfrak{R}$ of the set of all primes correspond sets S called $\mathfrak{R}$-tails that include with p all primes q for which $q\mathfrak{R}p$; and G is called $\mathfrak{R}$-dispersed if G is $\mathfrak{s}$-closed for each $\mathfrak{R}$-tail $\mathfrak{s}$. The first theorem which sharpens a previous result of the author [Math. Z. **71** (1959), 325–334; MR **21** #7250] is as follows: G is an $\mathfrak{r}$-closed group with $\mathfrak{R}$-dispersed $G/G_\mathfrak{r}$ if and only if G is an $\mathfrak{r}'$-homogeneous group whose $\mathfrak{r}'$-subgroups are all $\mathfrak{R}$-dispersed. The main theorem then reads as follows. The finite group G is $\mathfrak{r}$-closed if and only if G is $\mathfrak{r}'$-homogeneous and if each composition factor F of G satisfies at least one of the following conditions: (a) F is either an $\mathfrak{r}$-group or an $\mathfrak{r}'$-group. (b) $o(F)_\mathfrak{r}$ is the number of $\mathfrak{r}$-elements in F and $A \cap B = 1$ for any two distinct maximal $\mathfrak{r}$-subgroups A, B of F. (c) There exists a maximal $\mathfrak{r}'$-subgroup S of F such that $S \cap S^x = 1$ for each element x in F but not in $\mathfrak{N}S$. (d) $o(U)_\mathfrak{r}$ is the number of $\mathfrak{r}$-elements of U for each subgroup U of F, and all $\mathfrak{r}$-subgroups of F are nilpotent. (e) There exists an ordering $\mathfrak{R}$ of primes such that each $\mathfrak{r}'$-subgroup of F is also $\mathfrak{R}$-dispersed. Two corollaries follow the proof and discussion of the main theorem. *J. S. Frame* (E. Lansing, Mich.)

12141:

Fuchs, L. On the automorphism group of abelian p-groups. Publ. Math. Debrecen **7** (1960), 122–129.

Let G be an Abelian p-group, and define, as usual, $p^\alpha G$ for ordinals α. Denote by A the automorphism group of G, and denote by A_α the set of all automorphisms leaving $p^\alpha G$ elementwise fixed. A_α is a normal subgroup of A. The author first gives a new proof to a lemma of Zippin [Ann. of Math. (2) **36** (1935), 86–99]: Any automorphism of $p^\alpha G$ is induced by some automorphism of G. Let $A_\alpha{}^*$ be the set of all automorphisms leaving $p^{\alpha+1}G$ and $(p^\alpha G)[p] = \{x \mid x \in p^\alpha G \text{ and } px = 0\}$ both elementwise fixed, and let $A_\alpha{}^{**}$ be the set of all automorphisms leaving the

cosets of $(p^\alpha G)[p]$ mod $(p^{\alpha+1}G)[p]$ invariant. Then $A_\alpha \subseteq A_\alpha^* \subseteq A_\alpha^{**} \subseteq A_{\alpha+1}$, and A_α^*, A_α^{**} are both normal subgroups of A. The factor groups A_α^*/A_α and A_α^{**}/A_α^* are elementary abelian p-groups, and are completely determined when α is a non-negative integer or G is countable. The factor group $A_{\alpha+1}/A_\alpha^{**}$ is isomorphic to a subgroup of the general linear group $\mathrm{GL}(U_\alpha, \mathrm{GF}(p))$, where U_α is the rank of the factor group $(p^\alpha G)[p]/(p^{\alpha+1}G)[p]$. It is isomorphic to $\mathrm{GL}(U_\alpha, \mathrm{GF}(p))$ itself when α is a non-negative integer or G is countable. Finally, the author makes some remarks about the condition for solvability of the automorphism group A. *K. Honda* (Tokyo)

12142:

Sirotin, Yu. I. Anisotropic tensors. Dokl. Akad. Nauk SSSR **133** (1960), 321–324 (Russian); translated as Soviet Physics. Dokl. **5** (1961), 774–777.

A method is given of constructing generating systems for tensors of general rank which are invariant under coordinate transformations specifying one or other of the crystallographic groups. *J. E. Adkins* (Providence, R.I.)

12143:

Conrad, Paul. Some structure theorems for lattice-ordered groups. Trans. Amer. Math. Soc. **99** (1961), 212–240.

The main purpose of this paper is the study of l-groups which can be constructed from o-groups (i.e., linearly ordered groups) by forming cardinal sums and lexico-extensions. Definitions and notations: L is an l-group. $s \in L$ is basic if $0 < s$ and the set $\{x \in L\colon 0 < x \leqq s\}$ is ordered. Let $S \subset L$. S is a basis for L if S is a maximum set of disjoint elements and each $s \in S$ is basic. A disjoint set S is independent if each $s \in S$ is basic. $[S]$ is the subgroup of L generated by S. Condition (F): Each $x \in L$, $x > 0$, is greater than at most a finite number of disjoint elements. Condition (*): Each $x \in L$, $x > 0$, exceeds at least one basic element. L is a lexico-extension of S ($L = \langle S \rangle$) if S is an l-ideal of L and $x \in L$, $x \notin S$, $x > 0$, $y \in S \Rightarrow x > y$. Let $\{A_t\}$ be a finite set of o-groups. L is a lexico-sum of A_t if L can be constructed from the A_t by a finite alternating sequence of cardinal summations and lexico-extensions (each A_t is used exactly once to make a cardinal summation; the o-groups used to make the lexico-extensions may be arbitrary). If $\{A_t\}$ is an infinite set, the author defines in an analogous and natural manner the so-called small lexico-sum of o-groups A_t. L has finite rank if it contains only a finite number of convex subgroups; a lexico-extension $L = \langle S \rangle$ is said to be of finite rank if L/S is of finite rank. $N(L)$ is the set of all $s \in L$, $s > 0$, which are not weak units in L. Main results: L has finite rank if and only if L is a lexico-sum of $A_1, \cdots, A_n$, where each A_t has finite rank and the lexico-extensions used are also of finite rank. (This is a generalization of a theorem of G. Birkhoff [*Lattice theory*, Amer. Math. Soc. Colloq. Publ., Vol. 25, New York, 1948; MR **10**, 673; p. 237].) If A is a maximal convex chain of L that contains 0, then A is a convex o-subgroup of L. (This generalizes a result of the reviewer [Časopis. Pěst. Mat. **84** (1959), 53–63; MR **51** #3493].) L has a basis if and only if (*) holds; in this case a basis is a maximal independent subset of L. If (F) holds then L has a basis. Let $\{a_t\}$ be a maximal independent subset of L. For each a_t there exists a maximal o-subgroup A_t of L, $a_t \in A_t$. Then $[\bigcup A_t] = \sum + A_t$ (the small cardinal sum of o-groups A_t). L is a small lexico-sum of o-groups if and only if (F) holds. If L has a basis and if $F = \{0 < x \in L\colon$ x exceeds at most a finite number of disjoint elements$\}$, then $[F]$ is an l-ideal of L, $\sum + A_t \subset [F]$ and $[F]$ is the greatest convex subgroup of L that satisfies (F). If L satisfies (F), then L is a topological group in its interval topology if and only if L is ordered. L can be embedded in a large sum of o-groups if and only if L has a basis and no convex o-subgroup of L is bounded from above. L can be embedded in a large cardinal sum of subgroups of R if and only if L is archimedean and has a basis (R is the o-group of all real numbers). Theorem 9.1 (due to A. H. Clifford): $[N(L)]$ is an l-ideal in L; $N([N(L)]) = N(L)$; $L = \langle [N(L)] \rangle$; $[N(L)]$ is not a proper lexico-extension. In the last section it is supposed that L is commutative; let $\bar{L}$ be the divisible closure of L. Typical results: If $L = \langle B \rangle$, then $\bar{L} = \langle \bar{B} \rangle$; if L satisfies (F), then so does $\bar{L}$; from each basis for $\bar{L}$ a basis for L can be constructed; each basis for L is also a basis for $\bar{L}$.

J. Jakubík (Košice)

12144:

Jakubík, Ján. Principal ideals in l-groups. Czechoslovak Math. J. **9** (**84**) (1959), 528–543. (Russian. English summary)

Unter einem Hauptideal einer l-Gruppe versteht der Verfasser den Durchschnitt aller l-Ideale, die ein gegebenes Element der Gruppe enthalten. Wenn im System $S(G)$ aller Hauptideale der l-Gruppe G alle wachsenden Ketten endlich sind, so ist $S(G)$ ein vollständiger distributiver Verband. Ist die l-Gruppe G als Kardinalprodukt der l-Gruppen A, B darstellbar, und sind $S(G)$, $S(A)$, $S(B)$ Verbände, so ist $S(G)$ als Kardinalprodukt von $S(A)$ und $S(B)$ darstellbar. Ein ähnliches Resultat gilt auch für die sogenannten gemischten Produkte von l-Gruppen. Da jede Kette mit einem kleinsten Element zu dem System aller Hauptideale einer passenden abelschen l-Gruppe isomorph ist, so folgt daraus: Jeder Verband, der aus endlich vielen Ketten mit kleinsten Elementen durch Kardinalprodukt- und Ordinalsummenbildung in endlich vielen Schritten entsteht, ist zu dem System aller Hauptideale einer passenden abelschen l-Gruppe isomorph. Auf Grund dieser Resultate wird ein endlicher distributiver Verband charakterisiert, der zu dem System aller Hauptideale einer passenden abelschen l-Gruppe isomorph ist. Es ist ein endlicher distributiver Verband S mit einem kleinsten Element 0, der die folgende Eigenschaft hat: Sind x, y, z in S, $y \neq x \neq z$, $x = y \cup z$, so gibt es solche Elemente y_0, z_0 in S, dass $y_0 \leqq y$, $z_0 \leqq z$, $y_0 \cup z_0 = x$, $y_0 \cap z_0 = 0$ sind.

Für nichtisomorphe abelsche l-Gruppen G_1, G_2 können die Verbände $S(G_1)$, $S(G_2)$ zueinander isomorph sein. Die letzteren Ergebnisse stellen die Lösungen zweier Probleme von F. Loonstra dar [Compositio Math. **9** (1951), 130–140; MR **13**, 625]. In der Arbeit wird auch Birkhoffs Problem 99 [*Lattice theory*, rev. ed., Amer. Math. Soc., New York, 1948; MR **10**, 673; Chap. XIV, § 6] negativ gelöst.

M. Novotný (Brno)

12145:

Reiner, Irving. Behavior of integral group representations under ground ring extension. Illinois J. Math. **4** (1960), 640–651.

Let K be an algebraic number field and let R be a subring of K containing 1 and having quotient field K.

Let K' be a finite algebraic extension of K, and R' be a subring of K' having quotient field K' such that $R' \cap K = R$. The author gives some sufficient conditions under which R'-equivalence of R-representations of a finite group implies R-equivalence. The first main result is Theorem 2: If R is a principal ideal ring and two R-representations are both absolutely irreducible, then their R'-equivalence implies R-equivalence. Further, he shows that in case R is a valuation ring of a discrete valuation of K and R' is also the valuation ring of the extended valuation in K', and if their irreducible constituents in K are distinct from one another and absolutely irreducible, then R'-equivalence implies R-equivalence. For the proof he employs the useful concepts "i-intertwinable" and "everywhere intertwinable" and also the result due to J. M. Maranda [Canad. J. Math. **5** (1953), 344–355; MR **15**, 100]. By using this theorem and the result of P. Roquette [Arch. Math. **9** (1958), 241–250; MR **20** #3921], he also shows (Theorem 6): If G is a p-group, where p is an odd prime, and if R is the ring of p-integers of the rational field Q, K' a finite algebraic number field and R' any valuation ring of K' such that $R' \supset R$, then the R'-equivalent irreducible R-representations of G are also R-equivalent. *O. Nagai* (Yamaguchi)

12146:

Wallace, D. A. R. On the radical of a group algebra. Proc. Amer. Math. Soc. **12** (1961), 133–137.

Let G be a finite group of order $p^a m$, $(p, m) = 1$, and let Ω be an algebraically closed field of characteristic p. Denote by $N(G, \Omega)$ the radical of the group algebra $A(G, \Omega)$ of G over Ω. It is known that $\dim N(G, \Omega) \leqq p^a m - p^a m u_1^{-1}$, where u_1 is the degree of the indecomposable representation U_1 [Brauer and Nesbitt, Ann. of Math. (2) **42** (1941), 556–590; MR **2**, 309; p. 580]. It is stated that the equality $\dim N(G, \Omega) = p^a m - p^a m u_1^{-1}$ holds if and only if G has a normal p-Sylow group. But, as the author has communicated to the reviewer, the proof depends on the fact that the character η^1 of the representation U_1 is rational. Thus, the proof holds only for such groups, e.g., solvable groups and groups with a p-complement.

S. A. Amitsur (Jerusalem)

12147:

Morris, A. O. The spin representation of the compound and induced matrices of an orthogonal matrix. Quart. J. Math. Oxford Ser. (2) **12** (1961), 69–77.

The author derives explicit expressions for the basic spin character of the second induced and second compound matrices of an $n \times n$ orthogonal matrix A in terms of the characters of A. The results depend on whether n is even or odd. *H. K. Farahat* (Sheffield)

12148:

Munn, W. D. Pseudo-inverses in semigroups. Proc. Cambridge Philos. Soc. **57** (1961), 247–250.

The author begins by improving a result of M. P. Drazin [Amer. Math. Monthly **65** (1958), 506–514; MR **20** #5217; Theorem 7] to show that an element x of a semigroup S is pseudo-invertible (in Drazin's sense) if and only if some power of x lies in a subgroup of S. This result is then applied to show that a simple semigroup is completely simple if and only if it is pseudo-invertible. As a corollary it follows that in a pseudo-invertible semigroup every simple principal factor is completely simple.

G. B. Preston (Shrivenham)

12149:

Fantham, P. H. H. On the classification of a certain type of semigroup. Proc. London Math. Soc. (3) **10** (1960), 409–427.

A. H. Clifford [Ann. of Math. (2) **42** (1941), 1037–1049; MR **3**, 199] showed that a semigroup which is a union of groups can be mapped homomorphically onto a commutative semigroup of idempotents in such a way that the inverse image of each idempotent is a completely simple semigroup. The problem of determining in detail the structure of such a semigroup is thus reduced to investigating the relationship between its completely simple 'components'. For the case when the set of idempotents forms a commutative subsemigroup—when each completely simple component is a group—Clifford [loc. cit.] showed that products between the components could be determined and characterized by a set of homomorphisms. In the paper under review the author gives a similar but much more complicated treatment for the case when the set of idempotents forms a (not necessarily commutative) subsemigroup. In the characterization obtained the author makes no direct use of the components but uses instead homomorphisms between their subgroups. No light is thrown on the structure of semigroups of idempotents (further to that obtained by Clifford): the characterization obtained is in terms of semigroups of idempotents, groups and certain related homomorphisms.

G. B. Preston (Shrivenham)

12150:

Čupona, G. On reducible semigroups. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **11** (1958), 19–27. (Macedonian. English summary)

From the author's summary: "Let S be a semigroup. For $x \in S$ let $S(x)$ denote the set of all elements for which x is a right unit. Semigroups which admit decompositions of the form $S = \bigcup_{x \in S} S(x)$ and $S = \bigcup_{x \in S} S \cdot x$, the subsets $S(x)$ and $S \cdot x$, respectively, having some prescribed properties, are discussed. The purpose is to give a generalization of semigroups containing right units and to discuss an example of a semigroup in which the unique factorization theorem holds. Some definitions are formulated in more general terms." *Št. Schwarz* (Bratislava)

12151:

Čupona, G. On some relations between binary operations. Bull. Soc. Math. Phys. Macédoine **10** (1959), 5–27. (Macedonian. English summary)

From the author's summary: "Let $(S, \cdot)$ be a groupoid. By means of left translations a new operation $(+)$ is introduced into S. The relations between the two operations in $(S, \cdot, +)$ are studied under various suppositions concerning the groupoid $(S, \cdot)$. The relations obtained are of the following types: (1) $u(x+y) = u + xy$; (2) $u(x+y) = ux + y$; (3) $(u+v)(x+y) = uv + xy$; (4) $(u+v)(x+y) = ux + vy$."

Št. Schwarz (Bratislava)

12152:

Iséki, Kiyoshi; Sugita, Hajime. Commutativity of some continuous magnitude. Proc. Japan Acad. **36** (1960), 111.

A magnitude M is an additively written semigroup in which, for every pair $a, b \in M$, (1) $a, b < a+b$, (2) for $a \neq b$, either $a<b$ or $b<a$. (Here $a<b$ means that there is an $x \in M$ with $a+x=b$.) M is said to be continuous, if every bounded subset has a least upper bound. Theorem: Any continuous magnitude is commutative.

Št. Schwarz (Bratislava)

12153:

Yamada, Miyuki. Certain congruences and the structure of some special bands. Proc. Japan Acad. **36** (1960), 408–410.

Let S be an idempotent semigroup. It is known that there is a semilattice Γ and a disjoint family of rectangular subsemigroups S_γ such that $S = \bigcup\{S_\gamma \mid \gamma \in \Gamma\}$ and $S_\alpha S_\beta \subset S_{\alpha\beta}$, $\alpha, \beta \in \Gamma$.

For $\Delta \subset \Gamma$, define a relation $\mathfrak{R}_\Delta$ as follows: $a \mathfrak{R}_\Delta b \Leftrightarrow$ either $ab=a$ and both a, b are contained in the same S_γ, $\gamma \in \Delta$, or $ab=b$ and both a, b are contained in the same S_γ, $\gamma \notin \Delta$. A necessary and sufficient condition for $\mathfrak{R}_\Delta$ to be a congruence on S is given. Several new notions are introduced, but it is not possible to reproduce them here. Generalizations of the results of N. Kimura [same Proc. **33** (1957), 642–645; **34** (1958), 113–114, 121–123; MR **20** #4602, 4604] are given. All proofs are omitted.

Št. Schwarz (Bratislava)

12154:

Yamada, Miyuki. A note on subdirect decompositions of idempotent semigroups. Proc. Japan Acad. **36** (1960), 411–414.

Let S be an idempotent semigroup, $\mathfrak{R}_1, \cdots, \mathfrak{R}_n$ congruences on S, φ_i the natural homomorphism $\{S \to S/\mathfrak{R}_i\}$. Denote $S^* = \{\varphi_1(a), \cdots, \varphi_n(a) \mid a \in S\}$ by $S/\mathfrak{R}_1 \circ \cdots \circ S/\mathfrak{R}_n$.

For $i=1, 2, \cdots, n$ let $S_i = \bigcup\{S_\gamma^i \mid \gamma \in \Gamma\}$ in the sense of the preceding review, and $\mathfrak{D}_i$ the corresponding congruences on S_i. The set $S = \bigcup\{S_\gamma^1 \times \cdots \times S_\gamma^n \mid \gamma \in \Gamma\}$ is a subdirect product of $S_1, S_2, \cdots, S_n$ and is denoted by $S_1 \triangleright\!\triangleleft \cdots \triangleright\!\triangleleft S_n(\Gamma)$.

Now let $S = \bigcup\{S_\gamma \mid \gamma \in \Gamma\}$, $\mathfrak{D}$ the corresponding congruence on S, 0 the least element in the set of all congruences on S. The following assertion is proved: Let $\mathfrak{R}_1, \cdots, \mathfrak{R}_n$, $n \geq 2$, be congruences on S. Suppose that (a) $\mathfrak{R}_i \leq \mathfrak{D}$, (b) $\mathfrak{R}_1 \cap \cdots \cap \mathfrak{R}_n = 0$, (c) $(\mathfrak{R}_1 \cap \cdots \cap \mathfrak{R}_i)$ and $\mathfrak{R}_{i+1}$ are permutable, and (d) $(\mathfrak{R}_1 \cap \cdots \cap \mathfrak{R}_i) \cup \mathfrak{R}_{i+1} = \mathfrak{D}$ (for all i). Then

$$S \cong S/\mathfrak{R}_1 \triangleright\!\triangleleft \cdots \triangleright\!\triangleleft S/\mathfrak{R}_n(\Gamma) = S/\mathfrak{R}_1 \circ \cdots \circ S/\mathfrak{R}_n.$$

Št. Schwarz (Bratislava)

12155:

Lyapin, E. S. Representations of semigroups by partial mappings. Mat. Sb. (N.S.) **52 (94)** (1960), 589–596. (Russian)

By a representation of a semigroup $\mathfrak{A}$ by partial mappings of a set Ω we mean a homomorphism of $\mathfrak{A}$ into the set $\mathfrak{P}_\Omega$ of all partial mappings of Ω.

Let $\mathfrak{A} \subset \mathfrak{B}$ be semigroups, $L_2 \subset L_1$, $L_2 \neq L_1$, left ideals of $\mathfrak{B}$ (L_2 possibly the empty set) and $\Omega = L_1 \setminus L_2$. To every $a \in \mathfrak{A}$ we assign $\sigma(a)$ by the following definition. Consider the set Π_a of all elements $x \in \Omega$ for which $ax \in \Omega$ and define $\sigma(a)x = ax$ for $x \in \Pi_a$. ($\sigma(a)$ is a partial mapping of Ω with the domain Π_a.)

It is proved that $\sigma = \sigma(\mathfrak{A}, \mathfrak{B}, L_1, L_2)$ is a representation of the semigroup $\mathfrak{A}$ by partial mappings of the set Ω. Conversely, let $\mathfrak{S}$ be a given set of representations of the semigroup $\mathfrak{A}$ by partial mappings. Then there exists a semigroup $\mathfrak{B} \supset \mathfrak{A}$, such that for every representation $f \in \mathfrak{S}$ there are two two-sided ideals of $\mathfrak{B}$, $L_1^{(f)}$ and $L_2 = 0$ (the zero ideal of $\mathfrak{B}$), such that f is not essentially different (in a well defined sense) from $\sigma(\mathfrak{A}, \mathfrak{B}, L_1^{(f)}, 0)$.

Some special cases are explicitly stated.

Št. Schwarz (Bratislava)

12156:

Vagner, V. V. Semigroups associated with generalized heaps. Mat. Sb. (N.S.) **52 (94)** (1960), 597–628. (Russian)

This paper belongs to a series of papers by the author concerning an algebraic structure called "heap", which arose in connection with the study of inverse semigroups (called by the author "generalized groups"). It is self-contained, the previous results being reproduced at the appropriate places as needed.

Let (K, o) be a set with a ternary algebraic operation $o(k_1, k_2, k_3)$ defined for every triple $k_1, k_2, k_3 \in K$. Write $o(k_1, k_2, k_3) = [k_1 k_2 k_3]$. (K, o) is called a semiheap if (for $k_1, \cdots, k_5 \in K$)

$$[[k_1 k_2 k_3] k_4 k_5] = [k_1 [k_4 k_3 k_2] k_5] = [k_1 k_2 [k_3 k_4 k_5]]$$

holds. It is called a generalized heap if moreover (a) $[kkk] = k$ and (b) $[kk_1k_1k_2k_2] = [kk_2k_2k_1k_1]$, $[k_1k_1k_2k_2k] = [k_2k_2k_1k_1k]$ hold for every $k_1, k_2, k \in K$. To every generalized heap (K, o) we associate two idempotent semigroups (K, o_1), (K, o_2) by the definitions $o_1(k_1, k_2) = o(k_1, k_1, k_2)$, $o_2(k_1, k_2) = o(k_1, k_2, k_2)$.

In the first part of the paper several relations (subsets of $G \times G$) of an idempotent semigroup G are introduced and the connections between them are studied. These results are then applied to (K, o_1), (K, o_2). The relations in (K, o_1), (K, o_2) induce some relations in (K, o) (e.g., an order relation) and these are compared with relations introduced directly in (K, o). Also, a third groupoid (K, o_3), where $o_3(k_1, k_2) = o(k_1, k_2, k_1)$, is considered. A representation of (K, o) by one-to-one partial mappings of a (well defined) factor semigroup of (K, o_1) to a factor semigroup of (K, o_2) is given. If (G, o) is a generalized group, define a generalized heap $(G, \bar{o})$ by $\bar{o}(g_1, g_2, g_3) = g_1 g_2^{-1} g_3$. This enables one to apply the theory of generalized heaps to the study of generalized groups. The associated semigroups $(G, \bar{o}_1)$ and $(G, \bar{o}_2)$ are studied in detail, and a representation of (G, o) and $(G, \bar{o})$ by partial mappings is given.

In general, a number of rather deep results are obtained, but owing to the length of the formulation needed to explain them adequately, it is impossible to reproduce them here in detail.

Št. Schwarz (Bratislava)

12157:

Lefebvre, Pierre. Sur la plus fine équivalence simplifiable d'un demi-groupe. C. R. Acad. Sci. Paris **251** (1960), 1205–1207.

Soit D un demi-groupe. Une relation R est simplifiable à droite si $axRbx \Rightarrow aRb$. Soit E_d [E] l'ensemble des équivalences de D simplifiables à droite [à droite et à gauche]. La construction de l'élément nul σ_d de E_d (c'est-à-dire, l'équivalence simplifiable à droite la plus fine) est donnée. On prouve que l'équivalence σ_d est régulière à gauche. Puis la construction de l'élément nul τ de E est

donnée (τ n'est pas nécessairement régulière). Deux théorèmes concernant les demi-groupes stationnaires sont cités sans démonstrations. *Št. Schwarz* (Bratislava)

12158:

Lefebvre, Pierre. Sur la plus fine équivalence régulière et simplifiable d'un demi-groupe. C. R. Acad. Sci. Paris **251** (1960), 1265–1267.

L'équivalence régulière et simplifiable la plus fine Σ d'un demi-groupe D est construite.

En partant de l'égalité σ_1 (la plus fine équivalence régulière de D) nous construisons par récurrence une suite croissante de relations réflexives, symétriques et régulières $\sigma_1 \subseteq \sigma_2 \subseteq \cdots \subseteq \sigma_{2p-1} \subseteq \sigma_{2p} \subseteq \cdots$, telle que chacune de ces relations soit contenue dans toute équivalence régulière et simplifiable de D, celles d'indice impair étant des équivalences. L'auteur prouve: En prenant la réunion des relations σ_i, on a $\Sigma = \bigcup_{i \geq 1} \sigma_i$. *Št. Schwarz* (Bratislava)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 12529.

12159:

Gurevič, G. B.; Zalmanovič, Z. I. Orthogonally completable Lie algebras. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 75–97. (Russian)

Let A be a linear Lie algebra and L its orthogonal complement with respect to a Cartan metric. If L is also a Lie algebra, then A and L are called mutually orthogonal algebras. Let S be a common normalizer of the mutually orthogonal algebras A and L, and let T be the orthogonal complement of S. It is proved that T is a complete null-algebra contained in $A \cap L$; it is called the kernel of the algebra A. The algebra A is called orthogonally completable if $A+L$ is a Lie algebra. In this case, $A \cap L$ and $A+L$ are mutually orthogonal Lie algebras. It is proved that every orthogonally completable Lie algebra can be represented in the form $A = T+Z+M$, where T, Z, M are pairwise disjoint subalgebras, and further that T is the kernel of $A \cap L$, Z is commutative, $T+Z$ is the radical, and M is a maximal semi-simple subalgebra of A. The normalizer of an orthogonally completable Lie algebra is calculated. *A. L. Oniščik* (RŽMat **1959** #6682)

12160:

Vasil'ev, A. M. A class of orthogonal pairs of subgroups of the group O(N). Mat. Sb. (N.S.) **52 (94)** (1960), 917–946. (Russian)

Detailed exposition, with some extensions, of a result previously announced in Dokl. Akad. Nauk SSSR **121** (1958), 18–21 [MR **20** #4601].

Rimhak Ree (New Haven, Conn.)

12161:

Auslander, Louis. Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups. Ann. of Math. (2) **71** (1960), 579–590.

This paper generalizes some results of L. Bieberbach [Math. Ann. **70** (1911), 297–336; **72** (1912), 400–412] on space groups. The following is proved: "Let $G = N \cdot C$ be the semi-direct product of a connected, simply-connected nilpotent Lie group N and a compact Lie group C of automorphisms of N. Then for any discrete subgroup Γ of G with G/Γ compact, the coset space $N/(N \cap \Gamma)$ must be compact. Moreover, if Γ_1, Γ_2 are two such subgroups of G, then any isomorphism of Γ_1 onto Γ_2 can be extended to an automorphism of G." One notes that, when Γ is interpreted as a transformation group of N in the natural manner, then the assumption on Γ is equivalent to saying that Γ is totally discontinuous and has a compact fundamental domain. *H.-C. Wang* (Evanston, Ill.)

12162:

Palais, Richard S. A global formulation of the Lie theory of transformation groups. Mem. Amer. Math. Soc. No. 22 (1957), iii + 123 pp.

This memoir gives a modern account of Sophus Lie's theory of infinitesimal group actions on differentiable manifolds and related problems of integrability, both local and global. It should be noted that the author's differentiable manifolds are not necessarily Hausdorff.

Chapter I deals with foliations of a manifold given by involutive distributions [Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton, N.J., 1946; MR **7**, 412; p. 88]. A leaf of a foliation is a maximal connected integral manifold of the distribution, giving rise to a quotient space M/Θ. The author defines the notion of a coordinate system being regular with respect to a foliation by demanding that each leaf of Θ intersect the coordinate system in at most one m-dimensional slice (in the sense of Chevalley). He then gives definitions of a regular leaf and a regular foliation. Various theorems relating local and global structure of these notions are given and it is shown that the quotient space M/Θ by a regular foliation is a differentiable manifold.

In Chapter II the study of "infinitesimal" group actions (i.e., actions of a Lie algebra) and local group action on a manifold M is undertaken. After giving invariant definitions and theorems relating these two phenomena, the author's escape from coordinate systems is made good by considering the foliation defined on $G \times M$ by the involutive distribution spanned by all vectors of the form $(h_g, \rho(h)_x)_{(g,x)}$, where h is in the Lie algebra $\mathfrak{g}$ of left-invariant vector fields of G and ρ is the homomorphism defining the infinitesimal group action of $\mathfrak{g}$ on M. A detailed study of this foliation gives the first and second theorems of Lie.

Chapter III studies the problem of obtaining a global group action from a local group action and gives generalizations to Lie algebras of theorems in differential equations giving solutions for "all time". Using foliations the author obtains both necessary and sufficient conditions for global integration.

The main theorem of Chapter IV shows that if $X_1, \cdots, X_r$ are vector fields on M, each of which has integral curves defined for all time, and if they generate a finite-dimensional Lie algebra, then any linear combination (with constant coefficients) or bracket of them has all solutions defined for all time.

T. E. Stewart (Princeton, N.J.)

12163:

Hermann, Robert. C-W cell decompositions of symmetric homogeneous spaces. Bull. Amer. Math. Soc. **66** (1960), 126–128.

It is shown that if X is an infinitesimal motion on a symmetric space M, then the critical points of the length function $f(p)=\langle X_p, X_p\rangle$ on M are always nondegenerate critical manifolds. The author also gives a formula for the index of these critical manifolds in terms of the infinitesimal invariants of M and discusses the compact case in greater detail. *R. Bott* (Cambridge, Mass.)

12164:

Satô, Terukiyo. On linear Lie algebras of a certain dimension. Tôhoku Math. J. (2) **12** (1960), 71–76.

The author gives a proof of the theorem: In the Lie algebra $\mathfrak{sl}(n, K)$ (K being a field of characteristic zero) any proper subalgebra is at most of dimension n^2-n. The subalgebras of dimension n^2-n make up two conjugacy classes for $n \geqq 3$. *W. T. van Est* (Leiden)

12165:

Steinberg, Robert. Automorphisms of finite linear groups. Canad. J. Math. **12** (1960), 606–615.

In his fundamental paper in Tôhoku Math. J. (2) **7** (1955), 14–66 [MR **17**, 457], Chevalley takes a simple Lie algebra L over the complex field and shows that a Weyl basis of L may be chosen so that the structural constants are integers. Then he uses this basis to construct Lie algebras L_K over any field K. Let X_r, H_r be such a Weyl basis, where r runs through the roots of L with respect to a suitable Cartan subalgebra (to which the H_r belong). Chevalley shows that $\exp(\mathrm{ad}\, kX_r)$ makes sense as an automorphism of L_K, and then considers the group G of automorphisms of L_K generated by $\exp(\mathrm{ad}\, kX_r)$ as r runs through the positive and negative roots of L and k runs through K. "Most" of the groups so obtained are simple. The author, in Pacific J. Math. **9** (1959), 875–891 [MR **22** #79], adds to the list in Chevalley's paper by considering the subgroup of G generated by those $\exp(\mathrm{ad}\, kX_r)$ left fixed by a certain map which arises if L is of type D_4 or E_6 and K has an automorphism of order 3 or 2 respectively.

The author's purpose in this paper is to determine the automorphisms of these groups when K, whence also G, is finite. The automorphisms are built of four basic types: (1) inner automorphisms; (2) diagonal automorphisms; if h maps the additive group generated by the roots homomorphically into the multiplicative group of non-zero elements of K, then h induces a diagonal automorphism h^* of G such that $h^*\exp(\mathrm{ad}\, kX_r) = \exp(\mathrm{ad}\, h(r)kX_r)$; (3) field automorphisms; these arise because G is a linear group; (4) graph automorphisms; these arise from symmetries of the graph associated with the roots of L and also from some other automorphisms of the additive group generated by the roots.

The main results are: (1) Every automorphism α of G may be written in the form $\alpha = gfdi$, where g is a graph, f is a field, d is a diagonal and i is an inner automorphism; α determines g and f uniquely. (2) Let $\hat{G}$ be the group generated by G and the diagonal automorphisms, $\hat{A}$ the group generated by $\hat{G}$ and the group F of field automorphisms, and A the group of all automorphisms. Then $G \subseteq \hat{G} \subseteq \hat{A} \subseteq A$ is a normal sequence for A. (3) $\hat{G}/G$ is abelian. In fact, it is cyclic unless G is of type D_l (l even and the number of elements of K odd). A formula is given for the order of $\hat{G}/G$. (4) $\hat{A}/\hat{G} \approx F$ and thus is cyclic if K is finite. (5) $A = \hat{A}$ "most of the time" and the exceptions are given. *G. Leger* (Cleveland, Ohio)

12166:

Hochschild, G. On the algebraic hull of a Lie algebra. Proc. Amer. Math. Soc. **11** (1960), 195–199.

Chevalley a défini les algèbres de Lie algébriques comme les sous-algèbres de Lie de $\mathfrak{gl}_n(K)$ (K corps de caractéristique quelconque) qui sont les algèbres de Lie d'un sous-groupe algébrique de $\mathrm{GL}_n(K)$; lorsque K est de caractéristique 0, il a montré que toute sous-algèbre de Lie de $\mathfrak{gl}_n(K)$ est contenue dans une plus petite algèbre de Lie algébrique. Sa démonstration est longue et touffue, et consiste à prouver que si $x \in \mathfrak{gl}_n(K)$ et si G_x est l'intersection des groupes algébriques dont l'algèbre de Lie contient x, alors x appartient à l'algèbre de Lie de G_x. L'auteur donne une démonstration simplifiée de ce résultat, dont l'idée est de montrer que G_x est le groupe algébrique défini par l'idéal des polynômes P tels que $P(\exp(tx))=0$ dans l'anneau des séries formelles $K[[t]]$. La démonstration est facile lorsque K est algébriquement clos, mais nécessite un raisonnement délicat de spécialisation dans le cas général. {Note du rapporteur: si on dispose de la théorie des groupes formels, l'existence du plus petit groupe algébrique contenant un sous-groupe formel est presque triviale (en toute caractéristique) [cf. Dieudonné, Amer. J. Math. **79** (1957), 331–388; MR **20** #931; p. 367]; en caractéristique 0, on passe aisément de là aux algèbres de Lie grâce à la correspondance biunivoque bien connue entre algèbres de Lie et groupes formels dans ce cas.} *J. Dieudonné* (Paris)

12167:

Zassenhaus, Hans. On trace bilinear forms on Lie-algebras. Proc. Glasgow Math. Assoc. **4**, 62–72 (1959).

The author utilizes his results on Lie algebras of prime characteristic [Abh. Math. Sem. Univ. Hamburg **13** (1939), 1–100], over algebraically closed fields, and, in particular, his theorems on the representations of nilpotent Lie algebras, to study the structure of a Lie algebra L over a field F of characteristic $\neq 2, 3$ with relation to the trace form of a matrix representation of L. Let $g(x, y)$ denote one such form on L. Then if A is any solvable ideal in L, $[LA]$ is shown to be contained in $L^\perp$, the radical of the form g. If the representation is irreducible and if $g \neq 0$, then $[LA]$ lies in the kernel of the representation. It is next shown that there is a subalgebra U of L and a completely reducible representation ψ of U with $U + L^\perp = L$ (and $U \cap L^\perp$ is nilpotent), and such that $\mathrm{Tr}(\psi(a)\psi(b)) = g(a, b)$ for all $a, b \in U$. The representation ψ is the direct sum of certain of the irreducible constituents of the restriction to U of the original representation of L. The pair (U, ψ) has the further properties that

$$\psi([U, U^\perp(\psi)]) = 0 = \psi([UA]),$$

where $U^\perp(\psi)$ is the radical of the trace form of ψ on U, and A is an arbitrary ideal of U such that $\psi(A)$ is solvable. The main result of the paper is the "Structure Theorem": Let $\bar{L} = L/L^\perp$; then $\bar{L}$ carries a non-singular bilinear form induced by g, and $\bar{L}$ is a direct sum of indecomposable ideals $\bar{L}_i$ which are orthogonal with respect to this form. The ideals $[\bar{L}_i, \bar{L}_i] = \bar{L}_i^2$ are unique up to order, and $[\bar{L}_i^2, \bar{L}_i^2] = \bar{L}_i^2$. Furthermore, the center of $\bar{L}_i$ and $\bar{L}_i/\bar{L}_i^2$ have the same dimension. If $\bar{L}_i$ has center 0, then $\bar{L}_i$ is simple. If the characteristic is prime, and only then, there can be non-abelian $\bar{L}_i$ which are not simple; these have the property that $\bar{L}_i^2$ is the sum (not direct) of the

minimal non-vanishing "perfect" ($[AA]=A$) ideals of L contained in $\bar{L}_i$, and these are finite in number. If they are $L_{i1}, \cdots, L_{is}$, then L_i^2 contains the center $z(\bar{L}_i)$ of $\bar{L}_i$, and $\bar{L}_i^2/z(\bar{L}_i)$ is the direct sum of the simple non-abelian ideals $(L_{ij}+z(\bar{L}_i))/z(\bar{L}_i)$. The L_{ij} exhaust the minimal perfect ideals of L, where $L_{ij}=L_i$ if and only if $z(\bar{L}_i)=0$, and they are pairwise orthogonal with respect to the form on $\bar{L}$ induced by g. The classical example is that where L is the set of $n\times n$ matrices over a field F of prime characteristic dividing n, with $g(a,b)=\mathrm{Tr}\,(ab)$. Here $L^\perp=0$, so $\bar{L}=L$; further, L is indecomposable, so $\bar{L}=\bar{L}_1$ is the decomposition of the Structure Theorem. Now L^2 is an ideal of codimension 1 in L, and this is the dimension of $z(L)\subseteq L^2$ (for L^2 is the matrices of trace 0, $z(L)$ the scalar matrices). L^2 is perfect and is a minimal perfect ideal, since $L^2/z(L)$ is simple. Thus there is just one L_{ij}, namely L^2, and the simplicity of $L^2/z(L)$ corresponds to that of $(L_{ij}+z(\bar{L}_i))/z(\bar{L}_i)$ in the theorem. In the case where the form g itself is non-singular, i.e., where $\bar{L}=L$, one may compare the sharper results of R. Block [Proc. Amer. Math. Soc. **11** (1960), 377–379; MR **22** #5697] and the reviewer [Mem. Amer. Math. Soc. No. 19 (1956); MR **17**, 1108] under the assumption that the center of L is trivial.

G. B. Seligman (New Haven, Conn.)

12168:

Zassenhaus, Hans. On an application of the theory of Lie algebras to group theory. Proc. Sympos. Pure Math., Vol. 1, pp. 105–108. American Mathematical Society, Providence, R. I., 1959.

Recent work of the author makes feasible an analysis of the structure of Lie matrix algebras L over finite fields F such that the form $(a,b)=\mathrm{tr}\,(ab)$ is non-singular on L [see #12167; see also R. Block, Proc. Amer. Math. Soc. **11** (1960), 377–379; MR **22** #5697]. In this note, conditions are suggested under which these results and recent work on algebraic linear groups might be applied to yield information about certain matrix groups over F.

G. B. Seligman (New Haven, Conn.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See also 12058, 12401.

12169:

Hofmann, Karl Heinrich. Topologische Halbgruppen mit dichter submonogener Unterhalbgruppe. Math. Z. **74** (1960), 232–277.

Der Verfasser bezeichnet als submonogene Halbgruppen die (algebraisch) homomorphen Bilder eines induktiven Limes $G=\lim_{\to}(Z_i, p_{i+1}^i)$ einer abzählbaren Folge von Exemplaren Z_i der Halbgruppe Z der positiven ganzen Zahlen. Sind alle Inklusionsabbildungen p_{i+1}^i gleich der Multiplikation mit einer festen Primzahl p, so heißen die homomorphen Bilder von G scharf submonogen mit dem Teiler p. In § 1 werden die Möglichkeiten für Homomorphismen von G untersucht. In dem für das Verständnis des Hauptteils nicht nötigen § 2 werden verschiedene Topologisierungsmöglichkeiten für submonogene Halbgruppen betrachtet und miteinander verglichen. Solche Topologisierungen leiten sich aus den links offenen oder abgeschlossenen und rechts offenen oder unbeschränkten Intervallen bezüglich der natürlichen Anordnung von G ab. Ist G nicht monogen, d.h., sind die p_{i+1}^i nicht schließlich alle gleich 1, so gibt es auf der um 0 erweiterten Halbgruppe $\bar{G}$ genau eine feinste Topologie, die $\bar{G}$ zu einer topologischen Halbgruppe macht, und die echt feiner als die Intervalltopologie ist und bezüglich der die erzeugenden Elemente von G gegen 0 konvergieren. Die homomorphen Bilder von G werden stets mit der Quotiententopologie ausgestattet. Im dritten und letzten Paragraphen werden kompakte Halbgruppen K betrachtet, die höchstens einen idempotenten Ausnahmepunkt ∞ haben (der aber nicht vorhanden zu sein braucht), so daß $\underline{K}=K-\infty$ eine topologische Halbgruppe ist und die Multiplikation außer bei (∞,∞) überall stetig ist. Außerdem soll K für einen gewissen Homomorphismus f eine dichte submonogene Unterhalbgruppe fG enthalten. Bezeichnen G_r die Anfangsstücke und G^r die Endstücke von G bei der natürlichen Anordnung, so werden die Mengen $C=\bigcap_r \overline{fG_r}$ und $D=\bigcap_r \overline{fG^r}$ gebildet. $\underline{C}=C-\infty$ und $D-\infty$ sind leer oder lokal kompakte kommutative Gruppen mit kompakten Zusammenhangskomponenten. $C=\emptyset$ gilt genau dann, wenn G monogen ist. Wegen der Kompaktheit von K ist stets $D\neq\emptyset$. Je nachdem, ob $C=$ oder $\neq\emptyset$, ∞, K, und $D=$ oder $\neq\infty$ ist, erhält man verschiedene Typen von solchen Halbgruppen K. Für die möglichen Typen werden Beispiele konstruiert, und ihre Struktur wird in den einzelnen Fällen genauer beschrieben. Da die Formulierung des Hauptsatzes mehr als eine Druckseite in Anspruch nimmt, muß dafür auf die Arbeit selbst verwiesen werden. Genannt sei nur das folgende Ergebnis: Ist $\underline{C}\neq\emptyset$, so ist die $1\in\underline{C}$ auch Einselement von $\underline{K}$, und $\underline{C}$ ist die größte Untergruppe, die es enthält. Ist weiter G scharf submonogen mit dem Teiler p, so ist $\underline{C}$ kompakt und die Faktorgruppe von $\underline{C}$ nach der Zusammenhangskomponente ein projektiver Limes von endlichen monogenen (=zyklischen) Gruppen mit zu p teilerfremder Ordnung. Die Voraussetzung "scharf" ist dabei unentbehrlich.

H. Salzmann (Frankfurt a. M.)

12170:

Weiss, Edwin; Zierler, Neal. Locally compact division rings. Pacific J. Math. **8** (1958), 369–371.

Soient K un corps localement compact (non nécessairement commutatif), et μ une mesure de Haar de K additif; pour $a\in K$, l'automorphisme $x\to ax$ la transforme en une mesure proportionnelle $f(a)\mu$ ($f(a)\in\mathbf{R}$). Alors le "module" f est une valeur absolue sur K (à un exposant près), définissant la topologie donnée. D'où la classification bien connue des corps localement compacts. Tout ceci est classique.

P. Samuel (Clermont-Ferrand)

12171:

Aurora, Silvio. The embedding of certain metric fields. Michigan Math. J. **7** (1960), 123–128.

A theorem of Mazur states that a metric field K containing R (the real field), and whose norm N is such that $N(x)=|x|$ for $x\in R$, is either R or C (the complex field). [The terminology is that used in an earlier paper by the same author in Amer. J. Math. **80** (1958), 879–894; MR **20** #5192.] The author now examines the dropping of the hypothesis that $R\subset K$ and the weakening of the hypothesis about the norm by imposing it only upon a subset of the prime field of K. In each case the conclusion turns out to be that K is algebraically isomorphic to a subfield of C. Four sets of hypotheses are shown to imply

this conclusion, a typical set being that $K \supset P$ (the rational field) and $N(x) = |x|$ for all sufficiently small positive $x \in P$. The proofs are based upon a preliminary study of pseudonorms on general rings.

R. E. Edwards (Reading)

12172:

Aurora, Silvio. A note on certain connected metric division rings. Michigan Math. J. **7** (1960), 129–132.

In an earlier paper [Amer. J. Math. **80** (1958), 879–894; MR **20** #5192] the author generalised a theorem of Ostrowski by showing that if a connected metric field K has a norm N satisfying $N(x^2) = N(x)^2$ for all $x \in K$, then K is a division subring of the division ring Q of all quaternions. In the paper under review the same conclusion is shown to follow from weaker hypotheses, the multiplicative property of N being assumed to hold only on a portion of the division ring in question. A typical set of hypotheses is that K is a connected metric division ring which contains a set B with the following properties: (a) B is not nowhere dense in K; (b) $N(x_1 \cdots x_r) = N(x_1) \cdots N(x_r)$ for $x_1, \cdots, x_r \in B$; and (c) $N(\cdots xy \cdots) = N(\cdots yx \cdots)$ whenever $x \in B$. It is noted that in each case the assumption that K is a field would render unnecessary condition (c) and would lead to the conclusion that K is algebraically isomorphic to a subfield of the complex field. For this situation, see also the preceding related paper [#12171].

R. E. Edwards (Reading)

12173:

Varsavsky, Oscar A. Une topologie "taubérienne". C. R. Acad. Sci. Paris **250** (1960), 1951–1952.

The author defines a topology T in the space $\mathfrak{M}$ of all regular maximal ideals of a commutative ring A: if $a(M)$ denotes the natural projection of $a \in A$ in the field A/M, where $M \in \mathfrak{M}$, let E_a be the set of all $M \in \mathfrak{M}$ for which $a(M) = 1$, and let T be the weakest topology on $\mathfrak{M}$ in which all sets E_a are closed. The support of $a \in A$ is the T-closure of the set of all M such that $a(M) \neq 0$. Theorem: If A is semi-simple, and if I is an ideal in A which is not contained in any regular maximal ideal of A, then I contains all $a \in A$ whose T-support is a proper subset of $\mathfrak{M}$. This theorem is an analogue of the principal step in the proof of the abstract version of Wiener's Tauberian theorem in regular Banach algebras. If A has a unit, then T coincides with the hull-kernel topology. If A has no unit, then T is so weak that any two non-empty open sets intersect.

W. Rudin (Madison, Wis.)

12174:

Numakura, Katsumi. A note on Wedderburn decompositions of compact rings. Proc. Japan Acad. **35** (1959), 313–315.

A topological ring R with Perlis-Jacobson radical N is said to admit a Wedderburn decomposition if there is a closed subring S of R such that R is the module direct sum of S and N. It is shown that a compact ring R admits a Wedderburn decomposition if and only if every idempotent e of R/N can be raised to an idempotent e_1 of R such that for every neighborhood U of 0 in R there exist finitely many distinct primes $p_1, \cdots, p_k$ of R with $(p_1, \cdots, p_k)e_1$ in U. This generalizes a theorem of J. P. Jans [Duke Math. J. **24** (1957), 573–577; MR **19**, 939] who obtained such a necessary and sufficient condition in case N is open. Two corollaries are obtained; e.g., a compact ring with identity admits a Wedderburn decomposition if and only if it is the complete ring direct sum of algebras over finite fields.

M. Henriksen (Detroit, Mich.)

FUNCTIONS OF REAL VARIABLES

See also 12344, B12714, B12716.

12175:

Delachet, André. ★Les logarithmes et leurs applications. "Que sais-je?" Le Point des Connaissances Actuelles, no. 850. Presses Universitaires de France, Paris, 1960. 128 pp. Paperbound: 2.00 NF.

Assuming a knowledge of the elementary calculus on the part of the reader, the author defines the logarithm and derives its elementary properties, as well as those of the exponential function. Armed with this understanding, the reader is then shown how to use logarithms in practice.

12176:

Nevanlinna, F.; Nevanlinna, R. ★Absolute analysis. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 102. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vii+259 pp. DM 36; Ganzleinen, DM 39.

Le livre de R. et F. Nevanlinna sera certes le bienvenu: il est consacré à un exposé intrinsèque de l'analyse (les auteurs ont néanmoins soin de donner en traduction les propositions classiques correspondantes que fournit l'introduction des coordonnées).

Il ne s'agit pas cette fois d'une simple définition des notions fondamentales, mais aussi de leur utilisation pour la résolution des problèmes fondamentaux de la théorie des équations différentielles linéaires du premier ordre.

Le premier chapitre expose les notions fondamentales de la théorie des espaces vectoriels réels fini-dimensionnels et en particulier des espaces à produits scalaires et des espaces normés.

Le second chapitre introduit les concepts différentiels fondamentaux. Les auteurs utilisent les vectoriels munis de produits scalaires ou de normes. Le caractère affin des concepts fondamentaux résultera du fait que les définitions sont indépendantes du choix particulier des produits scalaires et des normes.

Suivant Stolz et Fréchet, une fonction f définie dans un ouvert du vectoriel V et à valeur dans le vectoriel W sera dite différentiable en x si l'on a

$$f(x+h) - f(x) = f'(x)h + |h|(h; x)$$

où, pour tout x de l'ouvert, $f'(x)$ désigne une application linéaire $V \to W$ et où $(h; x)$ désigne un vecteur de W qui tend vers 0 en même temps que $|h|$. L'application linéaire $f'(x)$ est appelée la dérivée de f et $f'(x)h$ sa différentielle. Ainsi la dérivée d'une fonction définie dans un ouvert du vectoriel V et à valeurs dans le vectoriel W est une application linéaire définie dans cet ouvert du vectoriel V et à valeurs dans l'espace $\mathcal{L}(V, W)$ des applications linéaires $V \to W$. Si cette application est différentiable, on pourra itérer le procédé et définir la dérivée f'' comme une

fonction définie dans l'ouvert de V et à valeurs dans l'espace des applications bi-linéaires $V \times V \to W$.

Sur ces bases, les auteurs introduisent la formule de Taylor et la différentiation partielle et ils exposent la théorie des fonctions implicites.

Le chapitre 3 introduit les différentielles alternées, leurs intégrales, la différentiation extérieure, la formule de Stokes pour un simplexe quelconque.

Le chapitre 4 est consacré à l'étude de l'équation $y' = f(x, y)$ où f désigne une fonction définie dans $V \times W$ et à valeurs dans $\mathscr{L}(V, W)$; la fonction inconnue est définie dans V et à valeurs dans W.

Le dernier chapitre est consacré aux éléments de géométrie différentielle et plus spécialement à l'étude des courbes et des hypersurfaces. On y retrouve les équations de Frenet et leur intégration, les notions classiques d'analyse tensorielle et le théorème de Gauss-Bonnet.

Comme ce livre est particulièrement "self-contained" il sera d'un usage aisé pour ceux qui veulent s'initier rapidement à ces théories fondamentales.

Si la langue est alerte et la rédaction fort soignée, on regrettera cependant un certain nombre d'abus de langage, hélàs courants en analyse, mais qui contribuent à obscurcir les notions fondamentales.

On regrettera également que la dualité entre dérivées et différentielles n'apparaisse pas explicitement.

G. Papy (Bruxelles)

12177:

Бермант, А. Ф. [Bermant, A. F.] ★Отображения. Криволинейные координаты. Преобразования. Формулы Грина [Mappings. Curvilinear coordinates. Transformations. Green's formulas]. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 306 pp. 5.85 r.

The material contained in the book constitutes what is generally thought of as half a course in advanced calculus, and is divided into four chapters: Chap. I discusses general properties of affine mappings and Jacobians; Chap. II deals with curvilinear coordinates; Chap. III entitled "Transformations of differential expressions", is devoted to a study of how differential expressions, such as the Laplacian, are transformed by mappings; while Chap. IV is devoted to the transformation of expressions containing integrals. *A. J. Lohwater* (Providence, R.I.)

12178:

Young, W. H. ★The fundamental theorems of the differential calculus. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 11. Hafner Publishing Co., New York, 1960. ix+72 pp. $3.00.

The original edition, of which this is a reprinting, appeared in 1910 [Cambridge Univ. Press, London].

12179:

Marcus, Solomon. Généralisation, aux fonctions de plusieurs variables, des théorèmes de Alexander Ostrowski et de Masuo Hukuhara concernant les fonctions convexes (J). J. Math. Soc. Japan **11** (1959), 171–176.

Let D be a convex subset of R^n and $f(x)$ a real function on D satisfying $f((x+y)/2) \leq (f(x)+f(y))/2$. If (*) $f(x)$ is bounded above at some $x_0 \in \mathrm{Int}\,(D)$, then $f(x)$ is continuous at each $y_0 \in \mathrm{Int}\,(D)$; for $f(x)$ is bounded above at $x_k = (x_{k-1}+z_0)/2$ $(k=1, 2, \cdots)$. Letting $z_0 = y_0 + (2^n-1)^{-1} \times (y_0 - x_0) \in D$, it follows that $f(x)$ is bounded above at y_0, hence continuous at y_0. (Let $y_0 = f(y_0) = 0$; thus, $f(2^{-1}x) \leq 2^{-1}f(x)$, $f(y) \geq -f(-y)$, etc.) (I) $f(x)$ is continuous in $\mathrm{Int}\,(D)$ if $f(x)$ is bounded above on a set $A \subset D$ such that $S^p(A)$ has an interior point for some positive integer p; here, $S^1(A) = A$ and $S^p(A) = \{z = (x+y)/2 : x, y \in S^{p-1}(A)\}$. After all, this implies (*). In particular, f is continuous when bounded above on a measurable set A of positive measure [for $n=1$, this is due to A. Ostrowski, Jber. Deutsch. Math. Verein. **38** (1929), Abt. 1, 54–62; p. 57]; for $(x+y)/2$ is interior to $S^2(A)$ when x, y are points of density for A. (II) $f(x)$ is bounded below on D whenever D is bounded and $f(x)$ is bounded below on a measurable set $A \subset D$ of positive measure (f may still be non-measurable, say, the square of an additive function). This is shown by using the special case $n=1$ due to M. Hukuhara [Proc. Japan Acad. **30** (1954), 683–685; MR **16**, 1007]. A simple proof of (II) is as follows. Let E be the non-empty set of points $x_0 \in \mathrm{Int}\,(D)$ such that each neighborhood $x_0 + U$ of x_0 contains a measurable set $x_0 + B$ of positive measure on which $f(x) > -K$ for some constant K. Clearly, E is closed relative to $\mathrm{Int}\,(D)$. Let $x_0 + 2U - U \subset D$ and let $\mu^*(C) > \mu(D) - \mu(2B)$, where $C = \{x \in D : f(x) < M\}$. If $y \in U$, then $x_0 + 2B - y$ and C have a point z in common; thus, $x_0 + y + z = 2w$ with $w \in x_0 + B$, hence, $f(x_0 + y) > -2K - M$. Therefore, E is open; thus, $E = \mathrm{Int}\,(D)$, in fact, $f(x)$ is bounded below at each $x_0 \in \mathrm{Int}\,(D)$, hence, at each $x_0 \in \mathrm{Cl}(D)$ (for, let v_0 be such that $x_0 + v_0 \in \mathrm{Int}\,(D)$, $x_0 + 2v_0 \in D$).

J. H. B. Kemperman (Madison, Wis.)

12180:

Marcus, S. Remarques sur les fonctions intégrables au sens de Riemann. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 433–439.

This paper treats a number of loosely related problems in the fine structure of functions in the class R of Riemann integrable functions. If $f \in R$, then f is bounded and continuous except on a set E of (Lebesgue) measure zero. The set E is further restricted, for it is of first category and is the union of a countable collection of sets of zero Jordan content. Frink [Ann. of Math. (2) **34** (1933), 518–526] showed that this rules out certain planar null sets of first category. The author gives a more general construction (ascribed to Erdös and corrected after publication by the author: the set A in the proof of Theorem 1 is perfect but need not have unit measure) which produces such examples in n-space for $n \geq 1$. Among other topics, the author discusses the relationship of R (which has cardinal number 2^c) to the sets of the Baire classification. There are functions in R which are exactly of class α, and there are Baire α functions which are not in R and which cannot be altered on a null set so as to be in R. If, however, two functions in R agree on a dense set, they agree a.e.; indeed, if both are derivatives, they agree everywhere. *R. C. Buck* (Princeton, N.J.)

12181:

Stoilov, S. [Stoilow, S.]. Continuous convergence. Rev. Math. Pures Appl. **4** (1959), 341–344. (Russian)

Let X be a metric space and $\{f_n(x)\}$ a sequence of real- or complex-valued functions defined on it. The sequence $\{f_n(x)\}$ is continuously convergent at the point $x_0 \in X$ if,

for every convergent sequence $\{x_\nu\}$, $x_\nu \in X$, $\nu = 1, 2, \cdots$, with $\lim_{n\to\infty} x_n = x_0$, there exists a number $\varphi(x_0)$, dependent only on x_0, such that $\lim_{n\to\infty} f_n(x_n) = \varphi(x_0)$. [The definition goes back to Hahn, *Reelle Funktionen*, Chelsea, New York, 1948, p. 222.]

The author proves the following. (1) If $\{f_\nu(x)\}$ is continuously convergent at every point of X, then its limit is a function continuous in X. (2) If X is compact, then continuous convergence implies uniform convergence. (3) In a metric space X, the necessary and sufficient condition for continuous convergence is uniform convergence in every compact subspace of X to a function which is continuous in X. (Essentially, (1) is Hahn's theorem 28.9.51 and (2) Hahn's theorem 28.9.4.) An example of a sequence of continuous functions which converge to a continuous function, but not continuously, is given.

V. Vučković (Belgrade)

12182:

Sion, Maurice. On integration of 1-forms. Pacific J. Math. **9** (1959), 277–286.

Pour définir une intégrale curviligne $\int_C \omega$, il n'est pas indispensable que la courbe C soit à variation bornée. Par exemple, dans le cas extrême (trivial) où ω est la différentielle d'une fonction f, l'intégrale peut être définie comme la différence des valeurs de f aux points extrêmes de C, et C n'est astreinte à aucune condition. H. Whitney [*Geometric integration theory*, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 309] a trouvé des espaces abstraits généraux de courbes pour lesquelles on peut définir l'intégrale de formes ω satisfaisants à certaines conditions. Dans le présent article, l'auteur considère les formes ω "possédents des dérivées jusqu'à un certain ordre", dans un sens un peu différent de celui qui était dû à Whitney [Trans. Amer. Math. Soc. **36** (1934), 63–89, 369–387] et avait été utilisé par lui-même précédemment [ibid. **77** (1954), 179–201; MR **16**, 344]. En ce qui concerne ω, il impose une condition de Lipschitz au reste d'un certain développement de Taylor; il définit d'autre part la variation α d'une courbe C comme la borne supérieure des sommes de puissances α de ses cordes. Il montre alors que l'intégrale existe si cette variation α est bornée, α étant en relation avec l'ordre maximum supposé des dérivées de ω. Sous quelques hypothèses restrictives pour C, il démontre que l'intégrale est "antiderivée" de ω sur C.

M. Janet (Paris)

12183:

Belowska, L. Résolution d'un problème de M. Z. Zahorski sur les limites approximatives. Fund. Math. **48** (1959/60), 277–286.

A consequence of a theorem of W. H. Young asserts that for each real-valued function on a real interval the subset of this interval where this function's upper left limit exceeds its upper right limit is at most countable. Resolving in the negative Zahorski's question as to whether or not in this assertion "limit" may be replaced by "approximate limit", the author constructs a function whose upper left approximate limit exceeds its upper right approximate limit on a set of the cardinality of the continuum.

T. A. Botts (Charlottesville, Va.)

12184:

Matysiak, A. Sur les limites approximatives. Fund. Math. **48** (1959/60), 363–366.

L. Belowska, in the preceding paper [#12183], has exhibited a real-valued function on the real line such that this function's upper left approximate limit exceeds its upper right approximate limit on a set of the cardinality of the continuum. In this note it is shown that this set is of the first category.

T. A. Botts (Charlottesville, Va.)

12185:

Marcus, Solomon. Sur certains problèmes et théorèmes concernant la continuité et la dérivabilité des fonctions. Monatsh. Math. **64** (1960), 119–130.

The author answers certain questions that have been suggested to him through the literature or orally, and that are related to such concepts as "fonction restreinte" (restricted function) [B. Meyer, Amer. Math. Monthly **62** (1955), 29–30; MR **16**, 1092] and "fonction apparantée" (cliquish function) [H. P. Thielman, ibid. **60** (1953), 156–161; MR **14**, 628]. Typical theorems are the following: (1) There exists a real function f defined and continuous on $[0, 1]$ with $0 \leq f(x) \leq 1$ such that for every t in $[0, 1]$, the set of values $\{x : f(x) = t,\ 0 \leq x \leq 1\}$ is infinite and denumerable. (2) If f is a real continuous function defined in a given interval I and such that each value y of the range $f(I)$ of f is taken on by the function an infinite but denumerable number of times then every interval J contained in I will contain a subinterval on which f is strictly monotone; furthermore, there exists a closed set P which is nowhere dense in I and is such that the range, $f(P)$, of f on P is $f(I)$. (3) Let a and λ be two real numbers and let f be a real function which has a derivative on (a, ∞) such that $\lim_{x\to\infty} f(x) = \lambda$. Then $\lim_{x\to\infty} f'(x) = 0$ if the derivative f' is uniformly lower semicontinuous on (a, ∞). The author establishes these and a number of related results.

H. P. Thielman (Pasadena, Calif.)

12186:

Bullen, P. S. Construction of primitives of generalized derivatives with applications to trigonometric series. Canad. J. Math. **13** (1961), 48–58.

The J_2 integral was defined by R. D. James [Canad. J. Math. **2** (1950), 297–306; MR **12**, 94] by means of the second symmetric Riemann derivative. In this paper J_3 and J_4 integrals are defined by use of third and fourth Riemann derivatives. These integrals are shown to have—like the J_2 integral—applications to summable trigonometric series.

For any function $F(x)$ let

$$H_3(F : x_1, x_2, x_3, x_4) = V_3(F : x_1, x_2, x_3, x_4)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3),$$

where V_3 is the third divided difference of F. Also let

$$D^3F = \lim_{h\to 0} \{F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h)\}/8h^3,$$

when this limit exists. The author defines the J_3 integral as follows. If $f(x)$ is finite and measurable on (a, b) and if there exists a function $F(x)$, continuous on $[a, b]$ and differentiable in (a, b), such that $D^3F = f$ in (a, b),

$$\int_{x_1,x_2,x_3}^{x} f(t)\, d_3t = H_3(F : x_1, x_2, x_3, x),$$

where x_1, x_2, x_3, x are any points in $[a, b]$ such that

$x_1 \leq x_2 \leq x_3$. Lemma 3.3 ensures the uniqueness of this definition.

Theorems 1 and 2 apply the J_3 integral to trigonometric series. If $\sum_{-\infty}^{\infty} c_n e^{int}$ is everywhere $(c, 1)$-summable to $f(t)$, then f is J_3 integrable and

$$c_n = -\frac{3}{8\pi^2} \int_{-4\pi,-2\pi,2\pi}^{0} f(t) e^{-int}\, d_3 t.$$

The J_4 integral is defined analogously to the J_3 integral. It is able to deal with $(c, 2)$-summable series $\sum_{-\infty}^{\infty} c_n e^{int}$ when $c_n = o(n)$ (Theorems 4 and 5).

H. Burkill (Sheffield)

12187:

Beesack, Paul R. Hardy's inequality and its extensions. Pacific J. Math. **11** (1961), 39–61.

The generalized form of Hardy's inequality is the following. If $p > 1$, $f(x) \geq 0$, $r \neq 1$, and $F(x)$ is defined as $\int_a^x f(t)dt$ or $\int_x^\infty f(t)dt$ according as $r > 1$ or $r < 1$, then

$$\int_0^\infty x^{-r} F^p\, dx < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf)^p\, dx,$$

unless $f \equiv 0$. In this paper more general inequalities relating $\int_a^b r(x) f^p dx$ and $\int_a^b s(x) F^p dx$ (where F is a suitable integral of f) are obtained and p is no longer restricted to the range $(1, \infty)$. The approach is via differential equations, a typical theorem (3.1.1) being the following.

Let $p > 1$ and suppose $f_1(x) \geq 0$, $r(x) > 0$ and $r(x)$, $r'(x)$, $s(x)$ are continuous in (a, b). Suppose also that r is such that $\int_a^b r f_1^p dx < \infty$ and, as $x \to a+$, $r(x) = O[(x-a)^{p-1}]$ or $r^{q/p}(x) \int_a^x r^{-q/p} dt = O(x-a)$, where $p^{-1} + q^{-1} = 1$. If, now, the differential equation

$$(d/dx)\{r(x) y'^{p-1}\} + s(x) y^{p-1} = 0$$

has a solution $y(x)$ such that $y(x) > 0$ and $y'(x) > 0$ in (a, b), then

$$\int_a^b s(x) F_1^p\, dx \leq \int_a^b r(x) f_1^p\, dx, \tag{1}$$

where $F_1(x) = \int_a^x f_1(t)dt$.

A similar theorem holds when $p < 0$; when $0 < p < 1$, the inequality in (1) is reversed. Analogous theorems are obtained with $F_2(x) = \int_x^b f_2(t)dt$. In all these theorems the question of equality is also considered. Hardy's inequality is shown to be easily obtained from Theorems 3.1.1 and 3.1.2, and several other inequalities, for particular r and s, are stated.

H. Burkill (Sheffield)

12188:

Nakamura, Masahiro. A remark on a paper of Greub and Rheinboldt. Proc. Japan Acad. **36** (1960), 198–199.

The author proves the following theorem: For $0 < m < M$, the inequality

$$\int_m^M t\, d\mu(t) \int_m^M \frac{1}{t}\, d\mu(t) \leq \frac{(M+m)^2}{4mM}$$

holds true for any positive Stieltjes measure μ on $[m, M]$ with $\|\mu\| = 1$. He then shows that this theorem is equivalent to a generalization of an inequality by Kantorovich recently published by W. Greub and this reviewer [Proc. Amer. Math. Soc. **10** (1959), 407–415; MR **21** #3774].

W. C. Rheinboldt (Syracuse, N.Y.)

12189:

Golovkin, K. K. Two classes of inequalities for sufficiently smooth functions of n variables. Dokl. Akad. Nauk SSSR **138** (1961), 22–25 (Russian); translated as Soviet Math. Dokl. **2**, 510–513.

If Φ_1 and Φ_2 are non-decreasing, left-continuous functions with unit variation on $[0, \infty]$ such that $\int_0^\infty p^{-1} d\Phi_1(p) = \int_0^\infty d\Phi_2(p)$, and u is a sufficiently smooth function on Ω in n-dimensional space, then for

$$\int_0^\infty \log \|u\|_{L_p}\, d\Phi_1(p) \leq \int_0^\infty \log \|u\|_{L_p}\, d\Phi(p),$$

it is necessary and sufficient that

$$\int_0^{\beta_1} p^{-1}\, d\Phi_1(p) + (\gamma_1/\beta_1)[\Phi_1(\beta_1)] \leq \int_0^{\beta_2} p^{-1}\, d\Phi_2(p) + (\gamma_2/\beta_2)[\Phi_2(\beta_2)]$$

implies

$$\int_0^{\beta_1} d\Phi_1(p) + \gamma_1[\Phi_1(\beta_1)] = \int_0^{\beta_2} d\Phi_2(p) + \gamma_2[\Phi_2(\beta_2)] \quad (\text{for } 0 \leq \gamma_i \leq 1).$$

If $\int_0^1 \alpha\, d\Phi_1(\alpha) = \int_0^1 \alpha\, d\Phi_2(\alpha)$, then for

$$\int_0^1 \log \|u\|_{\mathrm{Lip}\,\alpha}\, d\Phi_1(\alpha) \leq \int_0^1 \log \|u\|_{\mathrm{Lip}\,\alpha}\, d\Phi_2(\alpha),$$

it is necessary and sufficient that

$$\int_0^{\beta_1} \alpha\, d\Phi_1(\alpha) + \gamma_1\beta_1[\Phi_1(\beta_1)] \geq \int_0^{\beta_2} \alpha\, d\Phi_2(\alpha) + \gamma_2\beta_2[\Phi_2(\beta_2)]$$

implies

$$\int_0^{\beta_1} d\Phi_1(\alpha) + \gamma_1[\Phi_1(\beta_1)] = \int_0^{\beta_2} d\Phi_2(\alpha) + \gamma_2[\Phi_2(\beta_2)] \quad (\text{for } 0 \leq \gamma_i \leq 1).$$

S. Hoffman (Hartford, Conn.)

12190:

Král, Josef. Note on the Gauss-Ostrogradski formula. Časopis Pěst. Mat. **84** (1959), 283–292. (Czech. Russian and English summaries)

Let $A \subset E_{r+1}$ be a bounded set with closure $\bar{A}$ and boundary ∂A, and $S = \{\phi_n\}$ a countable collection of mappings of domains $G_n \subset E_r$ into $\bar{A}$. k is a fixed integer, $1 \leq k \leq r+1$. ϕ_{nj} $(j = 1, \cdots, r+1)$ are the components of ϕ_n, and $(\phi_{nj} : j \neq k)$ is assumed to map sets of measure zero into sets of measure zero. There are further conditions on the $\phi_n = \phi_n(t)$, assuring in particular the existence of the Jacobian J_n of $(\phi_{nj} : j \neq k)$ with respect to $t_1, \cdots, t_r$. It is further assumed that $\sum_n \int_{G_n} |J_n(t)|\, dt < \infty$, that distinct ϕ_n have disjoint ranges, and that these ranges cover ∂A except for a set whose projection on the coordinate hyperplane $x_k = 0$ has r-dimensional measure zero. It is proved that then ∂A itself has measure zero. Moreover, if for any set $T \in E_{r+1}$ and $x' = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{r+1}) \in E_r$, $T_{x'} = \{x_k \in E_1 : (x_1, \cdots, x_{r+1}) \in T\}$, then $(\bar{A})_{x'}$,

is a finite collection of compact intervals. Under these circumstances the author proves

$$\int_{E_r}\left(\int_{A_{x'}}\frac{\partial}{\partial x_k}f(x_1,\cdots,x_{r+1})\,dx_k\right)dx' = \sum_n(-1)^{k-1}\int_{G_n}f(\phi_n(t))J_n(t)\,dt,$$

provided f is continuous in $\bar{A}$ and, for almost every $x'\in E_r$, f is an absolutely continuous function of x_k on $(\bar{A})_{x'}$. The series on the right-hand side converges absolutely.

This result is used to derive Gauss' formula

$$\int_A \operatorname{div} v(z)\,dz = \sum_n\int_{G_n}v(\phi_n(t))\cdot w_n(t)\,dt$$

under very general assumptions. Here w_n is the vector whose kth component is $(-1)^{k-1}J_n$, $k=1,\cdots,r+1$.

A. Erdélyi (Pasadena, Calif.)

12191:

Arnol'd, V. I. On the representation of continuous functions of three variables by superpositions of continuous functions of two variables. Mat. Sb. (N.S.) **48 (90)** (1959), 3–74. (Russian)

In a note [Dokl. Akad. Nauk SSSR **114** (1957), 679–681; MR **22** #2668] the author deduced the first of the following theorems from the second and third, and he now gives the proofs of all three in full. (1) Every real continuous function $f(x_1, x_2, x_3)$ of three variables, given on the unit cube E^3, can be represented in the form

$$f(x_1,x_2,x_3)=\sum_{i=1}^{3}\sum_{j=1}^{3}h_{ij}[\phi_{ij}(x_1,x_2),x_3],$$

where the functions h_{ij} and ϕ_{ij} of two variables are real and continuous. (2) Every continuous function $f(x_1, x_2, x_3)$ given on E^3 can be represented in the form

$$f(x_1,x_2,x_3)=\sum_{i=1}^{3}h_i[\phi_i(x_1,x_2),x_3],$$

where h_i and ϕ_i are continuous functions and the functions h_i are real and defined on the product $\Xi\times E^1$ of a tree Ξ and the segment E^1, and the functions $\phi_i(x_1, x_2)$ are defined on the unit square and have points on Ξ as values. Here Ξ is a tree, the branching-index of the points of which is not greater than three. (3) Let F be any family of real equicontinuous functions $f(\xi)$, given on a tree Ξ, all points of which have a branching-index not greater than 3. Then the tree can be realized in the form of a sub-set X, homeomorphic to it, of the three-dimensional cube E^3 in such a way that every function f of the family F can be represented in the form

$$f(x)=\sum_{k=1}^{3}f_k(x_k),$$

where $x=(x_1, x_2, x_3)$ is the image of $\xi\in\Xi$ in the tree X, $f(x)=f(\xi)$, and the functions $f_k(x_k)$ of one variable are real and continuous and depend continuously on f in the sense of uniform convergence.

Theorem 1 contradicts a conjecture of Hilbert: for background and references see the author's survey [#12192]. Theorem 2, apart from its last sentence, is the special case $n=3$ of a result for functions of $n\geq 3$ variables given (with indications of the proof) by A. N. Kolmogorov [Dokl. Akad. Nauk SSSR **108** (1956), 179–182; MR **18**, 197]. For a tree Ξ with only two or three branches the conclusion of Theorem 3 is almost trivial, but the general case requires constructions and convergence arguments of very great complexity. The proofs of the lemmas leading to Theorem 2 are also very elaborate. However, the author's exposition is extremely clear and thorough and is illustrated by many admirable diagrams. He adds an appendix containing as much as he needs of A. C. Kronrod's theory [Uspehi Mat. Nauk **5** (1950), no. 1 (35), 24–134; MR **11**, 648] of the space of components of level-sets of a continuous function.

The author remarks that soon after he had finished his note, cited above, Kolmogorov [Dokl. Akad. Nauk SSSR **114** (1957), 953–956; MR **22** #2669] obtained a stronger result than Theorem 1, expressing $f(x_1, x_2, x_3)$ as the sum of seven terms each of the form $h[\phi_1(x_1)+\phi_2(x_2)+\phi_3(x_3)]$ with continuous h's and ϕ's the latter being independent of f.

H. P. Mulholland (Exeter)

12192:

Arnol'd, V. I. Some questions on approximation and representation of functions. Proc. Internat. Congress Math. 1958, pp. 339–348. (Russian) Cambridge Univ. Press, New York, 1960.

This is a very useful and readable survey of recent work on the problem of representing a function of n real variables by superposition of (continuous) functions of m real variables, with $m<n$, and on certain related approximation problems. Considering functions defined on an n-dimensional cube, the author denotes by C^n, $F_\infty{}^n$, or A^n the classes of functions that are continuous, infinitely often differentiable, or analytic, respectively; and he denotes by $F_{p,\alpha}{}^n$ the class of functions with pth derivatives all satisfying a Hölder condition of order α, where $0<\alpha\leq 1$. For any class X of functions let $\mathscr{S}(X)$ be the class of functions obtainable by superposition from a finite number of functions of X. The survey can now be summarized as follows. § 1. Hilbert's thirteenth problem (in effect, is A^3 contained in $\mathscr{S}(C^2)$?), and his theorem that A^3 is not contained in $\mathscr{S}(F_\infty{}^2)$. § 2. Superposition of smooth functions. A. G. Vituškin's result [*O mnogomernyh variaciyah*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; Dokl. Akad. Nauk SSSR **95** (1954), 701–704; MR **17**, 718; **15**, 945] that $F_{p,\alpha}{}^n$ is not contained in $\mathscr{S}(F_{q,\beta}{}^m)$ if $(p+\alpha)/n$ is less than $(q+\beta)/m$ and $q\geq 1$; an alternative to the complicated methods used by Vituškin, found by Kolmogorov [Uspehi Mat. Nauk **10** (1955), no. 1 (63), 192–194], using concepts drawn from information-theory. § 3. The "ε-entropy" of a "completely bounded" class X of functions; its alternative interpretation as the "minimum volume of a table" of a function in X "with accuracy ε". Bounds, of the order of $\varepsilon^{-n/(p+\alpha)}$, obtained by Vituškin [Dokl. Akad. Nauk SSSR **117** (1957), 745–747; **119** (1958), 418–420; MR **20** #3750; **21** #787] and by Kolmogorov [ibid. **108** (1956), 385–388; MR **18**, 324] for the ε-entropy of certain completely bounded sub-classes of $F_{p,\alpha}{}^n$; indication of the way in which these bounds can be used to prove Vituškin's result in § 2. § 4. Superposition of continuous functions. Positive results obtained by Kolmogorov and by the author by means of representations in two stages with a tree as "middle-space"; included in these, the positive answer, first obtained by the author to Hilbert's question (in § 1). (For details see the author's

memoir in Mat. Sb. (N.S.) **48 (90)** (1959), 3–74 [MR **22** #12191].) Improvement upon these results by Kolmogorov [Dokl. Akad. Nauk SSSR **114** (1957), 953–956; MR **22** #2669] who showed, by a more elementary method, not only that $C^n \subset \mathscr{S}(C^2)$, but, more precisely, that every function f in C^n is representable in the form

$$\sum_{i=1}^{2n+1} \chi^i\Big(\sum_{j=1}^{n} \phi_{ij}(x_j)\Big),$$

where the functions χ_i and ϕ_{ij} belong to C^1 and the ϕ_{ij}'s are independent of f. [It is a remarkable feature of this representation that the only function of two variables involved is their sum.] § 5. Problems. Tabulation of the known results and enunciation of several interesting outstanding problems. *H. P. Mulholland* (Exeter)

12193:

Boas, R. P., Jr. A series considered by Ramanujan. Arch. Math. **11** (1960), 350–351.

The function $f(x)$ defined by

$$xf(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(xe^{-x})^n, \qquad x > 1,$$

occurs in a problem set by Ramanujan [*Collected papers*, Univ. Press, London, 1927; p. 332; problem 738]. The reviewer and Auluck [J. Indian Math. Soc. (N.S.) **4** (1940), 169–173; MR **2**, 284] conjectured that $f(x)$ is completely monotonic for $x>1$, that is, that $(-1)^k f^{(k)}(x) \geq 0$ $(k=0, 1, 2, \cdots)$. The author shows that in fact $f(x)$ is not completely monotonic on any interval (c, ∞). There are minor misprints. *S. Chowla* (Boulder, Colo.)

MEASURE AND INTEGRATION

See also 12092, 12373, 12501, B12547, B12579.

12194:

Skvorcov, V. A. Interrelation between general Denjoy integrals and totalization $(T_{2S})_0$. Mat. Sb. (N.S.) **52 (94)** (1960), 551–578. (Russian)

Dans cet article l'auteur construit un exemple de fonctions $f(x)$ dont l'intégrale générale de Denjoy est différente de la totalisation $(T_{2S})_0$ [A. Denjoy, *Leçons sur le calcul des coefficients d'une série trigonométrique*, Gauthier-Villars, Paris, 1941; MR **8**, 260]. Pour $f(x)$, l'intégrale générale de Denjoy est égale à l'intégrale de Hinchin [Mat. Sb. (N.S.) **30 (72)** (1918), 543–557] et la totalisation $(T_{2S})_0$ est égale aux intégrales de Marcinkiewicz-Zygmund [Fund. Math. **26** (1936), 1–43], de James [Canad. J. Math. **2** (1950), 297–306; MR **12**, 94] et de Burkill [Proc. London Math. Soc. (3) **1** (1951), 46–57; MR **13**, 126]. De cet exemple, il resulte aussi que la totalisation $(T_{2S})_0$ et les intégrales de Marcinkiewicz-Zygmund, Burkill et James ne possèdent pas la propriété N de Luzin.

N. Dinculeanu (Bucharest)

12195:

Hartnett, W. E.; Kruse, A. H. Differentiation of set functions using Vitali coverings. Trans. Amer. Math. Soc. **96** (1960), 185–209.

Let $m, n, h, i, j = 1, 2, \cdots$, let X be a metric space, let s, t be the generic points of X, let A be a generic subset of X, let $\emptyset$ denote the void set, let $\mathscr{B}$ be the class of Borel subsets of X, let $m = m|\mathscr{B}$ be a measure on $\mathscr{B}$, let $\tilde{\mu} = \tilde{\mu}|\mathscr{B}^*$ be the completion of μ (μ is the Vitali covering of X) and $\mathscr{C}$ a subset of $\mathscr{B}^*$ such that: (1) $\emptyset \in \mathscr{C}$. (2) $0 < \tilde{\mu}(C) < \infty$ for each nonvoid $C \in \mathscr{C}$. (3) For each positive integer n the union of some countable subset of $\mathscr{C}_n = \{C | C \in \mathscr{C}, \text{ diam } (C) \leq n^{-1}\}$ is X. (4) If $\mathscr{D} \subset \mathscr{C}$, $\mathscr{D}_n = \mathscr{C}_n \cap \mathscr{D}$, $S_n =$ union of the $\mathscr{D}_n$-sets and $S = \bigcap S_n$, then there is a sequence $\{D_n\}$ in $\mathscr{D}$ such that $D_m \cap D_n = \emptyset$ for $m \neq n$ and $\tilde{\mu}(S - \bigcup D_n) = 0$. Let l be a non-negative, extended real-valued function defined on $\mathscr{C}$ such that $l(\emptyset) = 0$. $D^0(t, l)$ [resp. $D_0(t, l)$] = lim sup [resp. inf] $(l(C)/\tilde{\mu}(C))$ for $t \in C \in \mathscr{C}$, diam $(C) \to 0$. $D^{0*}(t, l)$ [resp. $D_0{}^*(t, l)$] = lim sup [resp. inf] $(l(C)/\tilde{\mu}(C))$ for $\emptyset \neq C \in \mathscr{C}$, diam $(C \cup \{t\}) \to 0$. Theorem 1: $D_0(\ , l)$ and $D^0(\ , l)$ are $\tilde{\mu}$-measurable. $\mu_n{}^*(A) = \inf\{\sum l(C_i) | C_i \in \mathscr{C}_n$ for each i, $A \subset \bigcup C_i\}$, $\mu^*(A) = \sup \mu_n{}^*(A)$. Let cov (A) be a system of non-void coverings $\mathscr{U}$ of A by open sets, directed by refinement. $l^0(\mathscr{U}) = \sup\{\sum l(C_j) | C_j \in \mathscr{C}$ and $C_j \subset U$ for some U in $\mathscr{U}$, $\tilde{\mu}(C_j \cap C_h) = 0$ for $j \neq h\}$, $\mu^{0*}(A) = \inf l^0(\mathscr{U})$. Theorems 2 and 3: $\mu_0{}^*$ and μ^{0*} are metric outer measures; therefore their restrictions to $\mathscr{B}$ are measures. Theorem 9: If μ is regular and $D_0(\ , l)$ finite-μ^* almost everywhere, there is a maximum measure ν on $\mathscr{B}$ such that, for each $B \in \mathscr{B}$, $\nu(B) \leq \mu_0{}^*(B)$ and $\mu(B) = 0$ implies $\nu(B) = 0$; $\nu(B)$ is the $\tilde{\mu}$-integral of $D^0(\ , l)$ on B. Theorem 15: If μ is regular and $\mu_0{}^*|\mathscr{B} \geq \nu$ for every μ-Lipschitzian measure such that $\mu^{0*}|\mathscr{B} \geq \nu$, then l is differentiable-μ^* almost everywhere. In Theorem 16 the results are specialized to the case in which l is a measure. The last section deals with the Lebesgue measure in R^q and uniformly regular closed intervals, q-simplices and compact sets as $\mathscr{C}$-sets. {Remarks by the reviewer: (1) The assertion $D^{0*}(t, l) = \max\{D^0(t, l), \text{lim sup } D^0(t, l)$ for $s \to t\}$, Remark 4, p. 188, is invalidated by the following example: $X = R^1$, $\tilde{\mu}$ is the Lebesgue measure, $\mathscr{C}$ is the set of closed intervals in R^1, $C_n = [(n+1)^{-1}, n^{-1}]$, $l(C_n) = \mu(C_n)$, $l(C) = 0$ if $C \neq C_n$ for each n. Hence $D^0(s, l) \equiv 0$ but $D^{0*}(0, l) = 1$. The μ^*-measurability of D^{0*} requires a new proof. Replacing "$l(C_n) = \mu(C_n)$" by "$l(C_n) = 1$" and letting $B = [0, 1]$ yields a counter-example for Theorem 12: $D^0(t, l)$ is namely $\equiv 0$, but $\mu^{0*}([0, 1]) = \infty$. The proof of Theorem 12, assuming $\tilde{\mu}$-measurability and finiteness everywhere of D^{0*}, is correct. (2) Misprints: Remark 9, line 5, read "$\nu(N)$ is finite". Lemma 4, line 2, read "$D_0(t, l)$". Proof of Theorem 14, read "13" instead of "14". Definition 12, line 3, read "$= 0$". (3) The reviewer is puzzled by the formulation of Theorem 13 since $\mu^0|\mathscr{B} = \mu^{0*}|\mathscr{B}$ and $\mu_0|\mathscr{B} = \mu_0{}^*|\mathscr{B}$ according to Definition 6 and Theorem 3. (4) The reviewer does not understand lines 7–8, p. 201. (5) Set functions similar to $\mu_0{}^*$ and μ^{0*} have been introduced as "Vitali integrals" by the reviewer, C. R. Acad. Sci. Paris **231** (1950), 1406–1408 [MR **12**, 487], by C. A. Hayes and the reviewer Canad. J. Math. **7** (1955), 221–274 [MR **17**, 719].}

Chr. Pauc (Nantes)

12196:

Mandl, Petr. Note sur les espaces compacts des mesures de probabilité. Časopis Pěst. Mat. **85** (1960), 133–140. (Czech and Russian summaries)

Let X be a topological space and $\mathscr{P}_0$ the space of Baire probability measures on X with weak topology. Let $\mathscr{P}_1$ be the space of regular Borel probability measures on X with weak topology. The author proves: If X is countably compact, then $\mathscr{P}_0$ is compact. If X is countably compact

and normal, then $\mathscr{P}_1$ is compact. The latter answers a question of the reviewer in Fund. Math. **38** (1951), 23–34 [MR **13**, 830]. The results are applied to the introduction of measures in product spaces.

J. H. Blau (Yellow Springs, Ohio)

12197:

Henstock, Ralph. The use of convergence factors in Ward integration. Proc. London Math. Soc. (3) **10** (1960), 107–121.

The author presents a generalized and simplified theory of the use of convergence factors in integration, combining convergence factors with Ward's integration. If the functions f, ϕ, F are defined and finite in $[a, b]$, F is termed a Ward major function of f, ϕ, if

$$[F(t)-F(x)-f(x)\{\phi(t)-\phi(x)\}]\operatorname{sgn}(t-x) \geqq 0$$

for $|t-x| \leqq \delta(x)$, $x \in [a, b]$. The author combines this idea of Ward, of using increments instead of derivatives, with the method of convergence factors and defines the so-called (N)-integral as follows. Let $N(x, h; t)$, $N(x, -h; t)$ be convergence factors defined respectively for $(0 \leqq t \leqq h,\ a \leqq x < x+h \leqq b)$, $(-h \leqq t \leqq 0,\ a \leqq x-h < x \leqq b)$. Suppose that one has defined the Stieltjes integration processes $(W)=(N_0), \cdots, (N_{r-1})=(S)$ beginning with Ward integration. F is termed an N-major function of f, ϕ if

$$(S)\int_{-h}^{0}\{F(x)-F(x+t)\}\,d_tN(x,-h,t) \geqq$$
$$f(x)(S)\int_{-h}^{0}\{\phi(x)-\phi(x+t)\}\,d_tN(x,-h;t)$$

for all intervals $(x-h, x)$ of a fixed family $\mathscr{L}$ of arbitrarily small left neighbourhoods of x, and similarly for $N(x, h; t)$ and a family $\mathscr{R}$ of right intervals, assuming that the stated (S)-integrals exist for the given ranges of x, h. The definition of the (N)-integral (the (N_r)-integral in the sequence) then follows the pattern of Perron's theory. The functions $N(x, \pm h; t)$ are called regular convergence factors for generalized Ward integration if: (a) $N(x, h; t-) \leqq N(x, h; t) \leqq N(x, h; t+)$, $N(x, h; 0) = N(x, h; 0+)$, and similarly for $-h$; (b) if ϕ is bounded, every Baire function that is a Ward major function of f, ϕ is also a N-major function of f, ϕ; (c) if F is an N-major function of 0, ϕ, then F is monotone increasing. Under these conditions the (N)-integral includes the Ward integral and preserves most of its properties. The author obtains necessary conditions for regularity and shows that they are sufficient in a special case that covers the theory of Jeffery and Miller [Duke Math. J. **12** (1945), 127–142; MR **6**, 204] which corresponds to the case $d_tN(x, h; t) = v(x, x+h, x+t)dt$.

M. Cotlar (Buenos Aires)

12198:

Henstock, Ralph. The equivalence of generalized forms of the Ward, variational, Denjoy-Stieltjes, and Perron-Stieltjes integrals. Proc. London Math. Soc. (3) **10** (1960), 281–303.

The (N)-integral introduced by the author in [#12197] is a generalization of the integral of Ward. Now he applies the same idea, of combining convergence factors with the use of increments, to define the N-Denjoy-Stieltjes integral, the N-Perron integral and the N-variational integral. The last one is the descriptive definition of the (N)-integral corresponding to the descriptive approach of Lusin, but using increments instead of derivatives. Given three functions H, f, ϕ, he constructs a function of intervals $c(x, h)$ (defined for each $(x, x+h)$ of the family $\mathscr{R}$) such that

$$c(x, h) = (S)\int_0^h [H(x+t)-H(x)-f(x)\{\phi(x+t)-\phi(x)\}]\,d_tN(x, h; t)|$$

and $c(x, -h)$ is defined similarly. The function H is termed N-variationally equivalent to f, ϕ, if one can choose $\mathscr{L}$, $\mathscr{R}$ and a monotone increasing function χ such that one has: (i) $0 = \chi(a) \leqq \chi(b) < \varepsilon$; and (ii) $c(x, h) \leqq (S)\int_0^h \{\chi(a+t)-\chi(x)\}d_tN(x, h; t)$, and similarly for $c(x, -h)$. If H is N-variationally equivalent to f, ϕ, then $H(b)-H(a)$ is called the N-variational integral of f, ϕ in $[a, b]$. H is said to be N-continuous with respect to f, ϕ, if $c(x; \pm h) \to 0$ as $h \to 0^+$. Let

$$w(u, v) = \sup_{0<h\leqq v-u}\Big[\,\Big|(S)\int_0^h \{H(u+t)-H(u)\}\,d_tN(u, h; t)\Big|;$$
$$\Big|(S)\int_{-h}^{0}\{H(v)-H(v+t)\}\,d_tN(v, -h; t)\Big|\,\Big].$$

Then H is termed an N-Denjoy-Stieltjes integral of f, ϕ if: H is N-continuous, every perfect set in $[a, b]$ contains a portion E such that ϕ is continuous and VB^* on E, f has a Lebesgue-Stieltjes integral with respect to ϕ on E, and $\sum w(u, v) < \infty$, $H(t)-H(x) = (LS)\int_{E\cap(x,t)} f\,d\phi + \sum_1 \{H(v)-H(u)\}$, where $\sum$ is extended over all (u, v) in $(a, b)-E$, $v-u < \delta$, and $\sum_1$ over all (u, v) in $(x, t)-E$. Similarly the N-Perron integral is defined. The author proves the equivalence of all these definitions of generalized integrals. In particular, as he observes, the proof shows that the Perron form does not really belong to the progression from the unconstructive Ward integral to the constructive Denjoy special integral, in that the taking of derivatives throughout is unnecessary.

M. Cotlar (Buenos Aires)

12199:

Kubota, Yôto. The Cauchy property of the generalized approximately continuous Perron integral. Tôhoku Math. J. (2) **12** (1960), 171–174.

The author proves [see also M. E. Grimshaw, Proc. Cambridge Philos. Soc. **30** (1934), 15–18] that the generalized approximately continuous Perron integral defined by C. Sunouchi and M. Utagawa [Tôhoku Math. J. (2) **1** (1949), 95–99; MR **11**, 90] has the following property: Let f be a real-valued function defined on the compact interval $[a, b]$. Suppose that f is integrable on $[a, x]$, for all $x \in [a, b)$, and let $F(x)$ be its primitive. If ap $\lim_{x\to b} F(x) = B$ exists, then f is integrable on $[a, b]$ and the integral of f on $[a, b]$ is equal to B. (It would be interesting to establish the exact relation between the above mentioned integral defined by G. Sunouchi and M. Utagawa and the approximately continuous Perron integral defined by J. C. Burkill [Math. Z. **34** (1931), 270–278].)

C. Ionescu Tulcea (Philadelphia, Pa.)

12200:

Albertoni, S.; Bocchieri, P.; Loinger, A. New theorem in the classical ensemble theory. J. Mathematical Phys. **1** (1960), 244–248.

An ergodic theorem is proved using the unitarity of the evolution operator, making certain assumptions about the geometric structure of the functional space considered, but not involving any hypothesis about metric transitivity.

D. ter Haar (Oxford)

12201:

Chacon, R. V. The influence of the dissipative part of a general Markov process. Proc. Amer. Math. Soc. **11** (1960), 957–961.

Let X be a space with a σ-field $\mathfrak{F}$ on it. The functions $\{P^n(x, A), n \geq 1, x \in X, A \in \mathfrak{F}\}$ are the transition probabilities of a general Markov process (σ-additive in $A \in \mathfrak{F}$, $\mathfrak{F}$-measurable in x) and so satisfy the Chapman-Kolmogorov equation. Let m' be a positive measure on $\mathfrak{F}$ such that the set of m'-absolutely continuous measures is mapped into itself by the transformation T, where $Tm(A) = \int_X P^1(x, A)m(dx)$. L_1 is the Banach space of all m'-integrable functions. A mapping L^* of L_1 into itself is given by $Tm''(A) = \int_A L^* f m'(dx)$ if $m''(A) = \int_A f m'(dx)$. E. Hopf [J. Math. Mech. **3** (1954), 13–45; MR **15**, 636] has shown that X splits into two disjoint sets C and D, $X = C + D$, the conservative and dissipative parts respectively, such that if $p_1 \in L_1$ is nonnegative, then $\sum_0^\infty L^{*k} p_1(x) < \infty$ for almost all $x \in D$, and such that if $p_2 \in L_1$ and is positive, then $\sum_0^\infty L^{*k} p_2(x) = \infty$ for almost all $x \in C$. Let $L_C{}^* f = e_C L^* f$ and $L_D{}^* f = e_D L^* f$, where e_C and e_D are the characteristic functions of C and D respectively. $f' = \lim_{n\to\infty} L_C{}^* \sum_0^{n-1} L_D{}^{*k} f$ exists almost everywhere and is in L_1. Let $L_n{}^*(f, p) = \sum_0^{n-1} L^{*k} f / \sum_0^{n-1} L^{*k} p$. Hopf asked whether the limit of $L_n{}^*(f - f', p)$ as $n \to \infty$ is zero almost everywhere on C. This paper gives a positive answer to this question.

M. Rosenblatt (Providence, R.I.)

12202:

Král, Josef. Closed systems of mappings and the surface integral. Czechoslovak Math. J. **10** (85) (1960), 27–67. (Russian. English summary)

Let $x = [x_1, \cdots, x_{r+1}] \in E_{r+1}$,

$$\pi_k x = [x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{r+1}],$$

$D = \{t\}$ a domain in E_r, T a mapping from D into E_{r+1}, $J(t, \pi_k T)$ the Jacobian of $\pi_k T$, $w_k(t, T) = (-1)^{k-1} J(t, \pi_k T)$ and $w(t; T) = [w_1(t, T), \cdots, w_{r+1}(t, T)]$. Let Λ be a countable system of indices, and, for every $\alpha \in \Lambda$, given a real number σ_α, a domain $D_\alpha \subset E_r$ and a locally Lipschitzian mapping T^α from D_α into E_{r+1} such that (*) $\sum_{\alpha\in\Lambda} (1 + |\sigma_\alpha|) \times \int_{D_\alpha} |w(t; T^\alpha)| dt < \infty$, and (**) $\Delta = \bigcup_{\alpha\in\Lambda} T^\alpha(D_\alpha)$ is bounded. For every bounded Borel function F on Δ, put $P_k(F; \mathfrak{S}) = \sum_{\alpha\in\Lambda} \sigma_\alpha \int_{D_\alpha} F(T^\alpha(t)) w_k(t; T^\alpha) dt$, where $\mathfrak{S}$ denotes the aggregate of triples $\{T^\alpha, D_\alpha, \sigma_\alpha\}_{\alpha\in\Lambda}$. The system $\mathfrak{S}$ is termed closed if (*) and (**) are satisfied and if $P(V; \mathfrak{S}) = \sum_{k=1}^{r+1} P_k(V; \mathfrak{S}) = 0$, whenever V is an infinitely differentiable vector satisfying $\operatorname{div} V = 0$ on E_{r+1}. For such a closed system $\mathfrak{S}$ the index at the point y is defined by $\iota(y; \mathfrak{S}) = (c_{r+1})^{-1} P(W^y; \mathfrak{S})$, where c_{r+1} is the area of the r-sphere, $W^y(x) = \operatorname{grad}_x \Phi(x, y)$, $\Phi(x, y) = (1-r)^{-1} |x-y|^{1-r}$ or $\log |x-y|$ according as $r > 1$ or $r = 1$ ($x \neq y$). The author establishes several properties of the index and surface integral of closed systems, and gives sufficient conditions for a system $\mathfrak{S}$ to be closed. For instance, fixing k, let $\pi_k T$ be denoted by $\hat{T}$, $\pi_k y$ by $\hat{y}$, $N(u, \hat{T})$ = the number of points in $\hat{T}^{-1}(u)$, $u \in E_r$, $\hat{T}_t(t) = x_t$ if $T(t) = x$, and let $y \in E_{r+1} - \bar{\Delta}$ with $\hat{y} \in E_r - \pi_k(\bar{\Delta} - \Delta)$, $\sum_\alpha N(\hat{y}, \hat{T}^\alpha) < \infty$. Then $2\iota(y; \mathfrak{S}) = \sum_{\alpha\in\Lambda} \sigma_\alpha \sum_t \mu_r(t; \hat{T}^\alpha) \cdot \operatorname{sgn} [T_k^\alpha(t) - y_k]$, where $\sum_t$ is extended over all t's in D_α with $\hat{T}^\alpha(t) = \hat{y}$, and $\mu_r(t; \hat{T}^\alpha)$ is the local index of $\hat{T}^\alpha$ at the point t. If the r-dimensional Lebesgue measure of $\pi_k(\bar{\Delta} - \Delta)$ is zero, then the measure of $\bar{\Delta}$ is zero, $\iota(y, \mathfrak{S})$ is defined a.e. in E_{r+1} and $\int_G \iota(y; \mathfrak{S}) \times \operatorname{div} V(y) dy = P(V; \mathfrak{S})$, where $G = \{y \in E_{r+1} - \bar{\Delta} : \iota(y; \mathfrak{S}) \neq 0\}$ and V is a vector on $G \cup \Delta$ fulfilling adequate conditions.

M. Cotlar (Buenos Aires)

FUNCTIONS OF COMPLEX VARIABLES

See also 12099, 12109, 12315, 12334, 12540, B12714, B13233.

12203:

Bieberbach, L. ★Einführung in die Funktionentheorie. 3te durchgesehene Aufl. Mathematische Leitfäden. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1959. 220 pp. DM 15.20.

The principal changes in this revision of the second edition [Verlag für Wissenschaft und Fachbuch, Bielefeld, 1952; MR **13**, 451] consist of those in the chapter on conformal mapping, which has been amplified by extra material, particularly in potential theory.

12204:

Meiman, N. N. On the theory of polynomials deviating least from zero. Dokl. Akad. Nauk SSSR **130** (1960), 257–260 (Russian); translated as Soviet Math. Dokl. **1**, 41–44.

Let $Q_n(z) = U_n(x, y) + iV_n(x, y) = z^n + c_1 z^{n-1} + \cdots + c_n$ be a polynomial, and let $A > 0$. Let $\mu(Q_n; A)$ denote the set of points for which the conditions $V_n(x, y) = 0$ and $|Q_n(z)| \leq A$ are satisfied, and $\mathfrak{G}(Q_n; A)$ the complement of $\mu(Q_n; A)$. Consider the function

$$\phi(z) = i \log \omega(z) \qquad (\operatorname{Re} \phi(-\infty) = -n\pi),$$

with $\omega(z) = A^{-1}(Q_n(z) + i\sqrt{(A^2 - Q_n{}^2(z))}$, where

$$\sqrt{(A^2 - Q_n{}^2(z))}/Q_n(z) \to i \qquad (z \to \infty).$$

The intersection of the region $\mathfrak{G}(Q_n; A)$ with the half-plane $\operatorname{Im} z > 0$ is denoted by $\mathfrak{G}_+(Q_n; A)$. Let ξ_i ($i = 1, 2, \cdots$), ξ_j ($j = 1, 2, \cdots$) be the zeros of $Q_n'(z)$; $\operatorname{Im} \zeta_i > 0$ ($i = 1, 2, \cdots$), $\operatorname{Im} \xi_j = 0$ and $\xi_j \notin \mu(Q_n; A)$ ($j = 1, 2, \cdots$). If the curves $\arg \omega(z) = \arg \omega(\xi_j)$ ($j = 1, 2, \cdots$) do not pass through other zeros of $Q_n'(z)$, then with the aid of the cuts along all the curves $\arg \omega(z) = \arg \omega(\zeta_i)$ ($i = 1, 2, \cdots$), $\arg \omega(z) = \arg \omega(\xi_j)$ ($j = 1, 2, \cdots$) there is obtained from $\mathfrak{G}_+(Q_n; A)$ a simply connected region $\mathfrak{F}(Q_n; A)$ with the boundary $\dot{\mathfrak{F}}$ (the circuit along $\dot{\mathfrak{F}}$ begins with $x = -\infty$). The function $\phi(z)$ can be represented in the form

$$\phi(z) = -\int_{z_\nu}^{z} \frac{Q_n'(z)}{\sqrt{(A^2 - Q_n{}^2(z))}} dz - (n - \nu + 1)\pi \qquad (z \in \mathfrak{F}(Q_n; A)),$$

where z_ν is the νth zero of the function $\sqrt{(A^2 - Q_n{}^2(z))}$ along $\dot{\mathfrak{F}}$, and the path of integration lies in $\mathfrak{F}(Q_n; A)$. The function $t = \phi(z)$ maps the region $\mathfrak{F}(Q_n; A)$ conformally onto a region $\mathfrak{T}(Q_n; A)$; this is an infinite half-strip

in the half-plane $\operatorname{Im} t \geqq 0$ having the real interval $[-n\pi, 0]$ as base with cuts parallel to the imaginary axis, and it is characterised by certain parameters determined by $Q_n(z)$ and A. It can be shown that to each half-strip $\mathcal{T}$ with a certain set of parameters there correspond a polynomial $Q_n(z)$, a constant A and a region $\mathcal{F}(Q_n; A)$. Two polynomials $Q_n^1(z)$, $Q_n^2(z)$ and constants A_1, A_2 correspond to half-strips having the same parameters if and only if $Q_n^2(z) = a^{-n} Q_n^1(az+b)$, $A_2 = a^{-n} A_1$ $(a>0)$. This investigation is based on the author's earlier results [same Dokl. **124** (1959), 1211–1214; MR **21** #2732].

K. Tandori (Szeged)

12205:

Meĭman, N. N. Polynomials deviating least from zero with an arbitrary number of given coefficients. Dokl. Akad. Nauk SSSR **130** (1960), 503–506 (Russian); translated as Soviet Math. Dokl. **1**, 72–75.

On the basis of his earlier results [#12204], the author obtains the polynomial $z^n + a_1 z^{n-1} + \cdots + a_{r-1} z^{n-r+1} + \cdots$, with the given leading coefficients $a_1, \cdots, a_{r-1}$, deviating least from zero on the interval $[-1, 1]$. He also gives the deviation of this polynomial from zero for arbitrary r.

K. Tandori (Szeged)

12206:

Sewell, W. E. The polynomial derivative at a zero angle. Proc. Amer. Math. Soc. **12** (1961), 224–228.

Let C be a piecewise analytic Jordan curve, z_0 a corner of C with exterior angle zero. We assume that the tangential derivatives up to the order t of the arcs adjacent to z_0 vanish at z_0. If $P_n(z)$ is a polynomial of degree n satisfying $|P_n(z)| \leq M$ on C, then $|P_n'(z_0)| \leq MK(\log n)^{1/t}$, where K depends only on C. For the special case of a cardioid it is shown that at the corner a bound of order $(\log n)^2$ is valid.

G. Szegö (Stanford, Calif.)

12207:

Squire, William. The Lienard-Chipart criteria for the stability of polynomials. J. Aerospace Sci. **28** (1961), 255–256.

This note in the Reader's Forum draws attention to the Lienard-Chipart criteria for a polynomial to have only roots with negative real parts. The author states that, while the Routh-Hurwitz criteria are well known, the improvements due to Lienard and Chipart "are not widely known and are being rediscovered by various investigators". For a full discussion the reader is referred to F. R. Gantmaher, *Teoriya matric* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; English translations, Interscience, New York; Chelsea, New York; 1959; MR **16**, 438; **21** #6372b, c].

A. M. Duguid (London)

12208:

Guha, U. C. Rearrangements of the coefficients of a power series. Math. Z. **75** (1960/61), 325–327.

Let π denote a permutation of the non-negative integers. The following results are proved. (i) A necessary and sufficient condition for the radius of convergence of any power series $\sum_0^\infty a_n z^n$ to be equal to the radius of convergence of $\sum_0^\infty a_{\pi(n)} z^n$ is $\pi(n) = n + o(n)$. (ii) A necessary and sufficient condition that, for any power series $\sum_0^\infty a_n z^n$ with radius of convergence 1, the radius of convergence of $\sum_0^\infty a_{\pi(n)} z^n$ should again be 1 is $\pi(n) = O(n)$. The argument proceeds from first principles, but it is also pointed out that the above results can be derived from the work of I. M. Sheffer [Duke Math. J. **11** (1944), 167–180; MR **5**, 236] and K. Zeller [Math. Nachr. **10** (1953), 175–177; MR **15**, 325].

L. Mirsky (Sheffield)

12209:

Pokornyĭ, V. V.; Rybin, P. P. Stabilization of a process for determining formal implicit functions. Uspehi Mat. Nauk **15** (1960), no. 4 (94), 169–172. (Russian)

The authors consider the classical problem of determining the solutions $\alpha = \alpha(\lambda)$ of an equation

$$(1) \qquad F(\alpha, \lambda) \equiv \sum_{j+k \geq 1} T_{jk} \alpha^j \lambda^k = 0$$

from the point of view of formal power series. The method (of undetermined coefficients) is to seek solutions of the form

$$(2) \qquad \alpha = \sum_{k=1}^{\infty} \alpha_k \lambda^{k/s} \qquad (s \geq 1),$$

substitute (2) into (1), and collect the coefficients P_n of λ^n in the result. These coefficients are then set equal to zero: (3) $P_n(\alpha_1, \cdots, \alpha_n) = 0$. From the conditions (3), the coefficients α_k together with the number s are to be found successively. According to a definition of M. A. Krasnosel'skiĭ, the process of successive determination of the coefficients α_k from the conditions (3) becomes "stabilized" if, for all n beginning with some number n_0, the equations (3) are linear with respect to α_n, with one or more of the coefficients of the terms involving α_n not equal to zero. The following theorems are obtained. (I) The process of formal construction of the solution of (1) always becomes stabilized. (II) The stabilization of the process of construction of the formal series (2) satisfying (1) takes place no later than the first non-zero coefficient to occur in the sequence $\{T_{0k}\}$. The results are applied to the study of non-linear integral equations of the form:

$$\varphi(z) = \int_0^1 A_{10}(x, y)\varphi(y)\, dy + \lambda \int_0^1 A_{01}(x, y)\, dy + \int_0^1 \left\{ \sum_{j+k \geq 2} A_{jk}(x, y)[\varphi(y)]^j \lambda^k \right\} dy,$$

using the method of Nekrasov-Nazarov.

D. H. Hyers (Los Angeles, Calif.)

12210:

Watson, G. N. ★Complex integration and Cauchy's theorem. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 15. Hafner Publishing Co., New York, 1960. vii + 79 pp. $3.00.

The original edition, of which this is a reprinting, appeared in 1914 [Cambridge Univ. Press, London].

12211:

Żakowski, W. Sur un problème non linéaire de Hilbert. Ann. Polon. Math. **9** (1960/61), 79–99.

Let L be a set in the complex plane consisting of finitely many Jordan curves and finitely many arcs. It is desired to find functions $\Phi_1(z), \cdots, \Phi_n(z)$ holomorphic in the complement of L and satisfying

$$\Phi_r^+(t) = G_r(t)\Phi_r^-(t) + \lambda F_r[t, \Phi_1^+(t), \cdots, \Phi_n^+(t), \Phi_1^-(t), \cdots, \Phi_n^-(t)] + g_r(t)$$

($\nu=1, \cdots, n$) on L. Here $\Phi_\nu^{\pm}$ denote the two boundary values of Φ_ν. The G_ν satisfy Hölder conditions, F_ν generalized Hölder-Lipschitz conditions, and g_ν Hölder conditions except near the end points of the arcs of L where certain types of discontinuities are allowed. The Φ_ν may be unbounded (but must satisfy certain inequalities) near these end points. The problem is equivalent to the solution of a system of singular integral equations on L. The existence of a solution for sufficiently small λ is proved using the Schauder fixed-point theorem. Under certain more restrictive conditions the method of successive approximation can be applied, and both existence and uniqueness of the solution are obtained. *H. Widom* (Ithaca, N.Y.)

12212:

Landau, H. J. On canonical conformal maps of multiply connected domains. Trans. Amer. Math. Soc. **99** (1961), 1–20.

Die Arbeit schließt an folgenden Satz von Walsh an [dieselbe Trans. **82** (1956), 128–146; MR **18**, 290]. Es sei D ein Gebiet der z-Ebene mit $\infty \in D$, welches von den paarweise fremden Jordankurven B_i ($i=1, \cdots, \mu$) und C_j ($j=1, \cdots, \nu$) berandet wird. Dann existiert eine konforme Abbildung von D auf ein Normalgebiet Δ der Z-Ebene, welches definiert wird durch

$$1 < |T(Z)| < e^{\alpha}, \quad T(Z) = A\,\frac{(Z-a_1)^{M_1}\cdots(Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1}\cdots(Z-b_\nu)^{N_\nu}},$$

wobei α, M_i, N_j gewisse positive Konstanten sind mit $\sum M_i = \sum N_j = 1$. Dabei geht $\bigcup B_i$ in $|T(Z)|=1$ und $\bigcup C_j$ in $|T(Z)|=e^{\alpha}$ über; die a_i werden durch $|T(Z)|=1$ und die b_j durch $|T(Z)|=e^{\alpha}$ von D getrennt. Zwei solche kanonische Abbildungen von D unterscheiden sich um eine lineare Transformation. Ein analoger Satz gilt, wenn die B_i in Punkte ausarten. Der Verfasser gibt in Teil I der Arbeit einen neuen Beweis dieses Satzes, welcher im Gegensatz zu den Beweisen von Grunsky [Math. Z. **67** (1957), 129–132; MR **19**, 538] und Jenkins [Trans. Amer. Math. Soc. **88** (1958), 207–213; MR **19**, 538], welche auf Uniformisierung beruhen, konstruktiver Natur ist. Schlüssel dazu ist folgender Satz: Zu jedem Ringgebiet D' (Innenrand C_1, Außenrand C) gibt es eine konforme Abbildung des ganzen Äußern von C_1 derart, daß das harmonische Maß des Bildes von C in das ganze Innere des Bildes von C_1 harmonisch fortgesetzt werden kann, mit Ausnahme einer logarithmischen Singularität. Beweis durch ein Iterationsverfahren, welches an das Koebesche Iterationsverfahren erinnert. In Teil II werden die genannten kanonischen Abbildungen durch Extremaleigenschaften charakterisiert, und $\alpha/2\pi$ als extremale Länge der Kurvenschar erkannt, die $\bigcup B_i$ mit $\bigcup C_j$ in D verbindet. Teil III verwendet die Methode von Teil I zur effektiven Ermittlung der konformen Abbildung eines Ringgebiets auf einen Kreisring. Das Verfahren besteht in der sukzessiven Abbildung von Außengebieten auf das Äußere von Kreisen (Normierung in ∞), und nachfolgender Spiegelung an diesen Kreisen. Es ist eng verwandt mit einem Verfahren von Komatu [Proc. Japan Acad. **21** (1945), 146–155 (1949); MR **11**, 341] und seiner Modifikation durch Einschaltung von Spiegelungen; siehe Gaier [Arch. Rational Mech. Anal. **3** (1959), 149–178, insbesondere § 9 C; MR **21** #3934]. Für die Konvergenzgeschwindigkeit erhält der Verfasser $|F(z)-F_n(z)| \leq M\rho^{2^n}$, wo M und $\rho<1$ vom Ringgebiet abhängen. Man vergleiche die mit funktionentheoretischen Methoden gewonnenen konkreten Abschätzungen bei Gaier [loc. cit. S. 171–173].

D. Gaier (Giessen)

12213:

Pommerenke, Christian. Über die Werte der Faberschen Polynome. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1959**, 273–277.

Let G be simply connected and contain $\zeta=\infty$; let $\zeta=\varphi(w)=cw+c_0+c_1w^{-1}+\cdots$ be the mapping onto $|w|>1$, $c>0$. The Faber polynomials $F_n(z)$ can be defined by

$$\frac{\varphi'(w)}{\varphi(w)-z} = \sum_{n=0}^{\infty} F_n(z)w^{-n-1}, \qquad |w|>1,$$

where z is in the complement of G. Let $M(a)$ be the derivative of the set of values z with $F_n(z)=a$. Denoting the boundary of G by R, the following two assertions of "Jentzsch type" are proved. (1) Let $a\neq 0$. Then $M(a)$ contains R. (2) The same holds for $a=0$ if $\varphi(w)$ has a singularity on $|w|=1$. If $\varphi(w)$ is regular on $|w|=1$, the set $R \cap M(0)$ coincides with that of the "multiple" points of R. A point $c \subset R$ is called a multiple point if the equation $\varphi(w)=c$ has more than one solution on $|w|=1$. (The number of these solutions is actually ≤ 2.)

G. Szegö (Stanford, Calif.)

12214:

Tietz, Horst. Zur Klassifizierung meromorpher Funktionen auf Riemannschen Flächen. Math. Ann. **142** (1960/61), 441–449.

Let G be a normal domain (i.e., G is the only relatively compact component of the complement of δG = boundary G) in the open Riemann surface X. Behnke and Stein [Math. Ann. **120** (1948), 430–461; MR **10**, 696] have proved the existence of a Cauchy kernel for δG. The author assigns to each f meromorphic on the closed set $X-G$ two functions $L_G f$ and $L_G{}^* f$ (analytic on G and on $X-G$, respectively) obtained by integrating the product of f with the Cauchy-Behnke-Stein kernel on a family of curves homologous with δG and restricting the parameter to lie in G and in $X-G$, respectively. On a neighborhood of δG the relation $L_G f + L_G{}^* f = f$ is satisfied by virtue of the Cauchy integral theorem. Thus the relation $L_G f = f - L_G{}^* f$ defines an extension of $L_G f$ which is meromorphic on all of X and consequently shows that $g = L_G{}^* f$ and f are meromorphic on $X-G$, with their difference being analytic on G. Moreover, $L_G g = 0$ from an easy computation. The author notes a converse: If any meromorphic g is so related to f, then $g = L_G{}^* f$ must follow.

The principal part of a function with a pole at z_0 is customarily a polynomial g in $1/(z-z_0)$, a polynomial with no constant term. This is revealed suggestively by the fact that $L_G g = 0$ for small neighborhoods G of z_0. The author makes the generalization: If $z_0 \in X$ is a pole of f (meromorphic on X), then a principal part of f at z_0 is any function g, meromorphic on X with $f-g$ analytic at z_0 and $L_G g = 0$ for all normal domains G containing z_0. By the converse mentioned above, there is just one principal part, namely, $g = L_G{}^* f$ for G any sufficiently small neighborhood of z_0.

An exhaustive system $\mathfrak{G}$ is a set, directed by inclusion, whose members are all normal domains, presumably covering X. If $\mathfrak{G}$ is an exhaustive system, the nets $\{L_G f\}_{G \in \mathfrak{G}}$, $\{L_G{}^* f\}_{G^* \in \mathfrak{G}}$ both converge compactly on

$X-\pi_f$ (π_f is the set of poles of f) or else both diverge. In the convergent case one has $F+g=f$, where F and g denote the nets' respective limits, and one writes $f\in\mathfrak{R}_{\mathfrak{G}}$. The F and g are analytic on $X-\pi_f$ and F is analytic on X. When $F=0$ so that $g=f$, one writes $f\in\mathfrak{R}_{\mathfrak{G}}{}^0$. With $f\in\mathfrak{R}_{\mathfrak{G}}$ one calls g a $\mathfrak{G}$-representation of the principal parts of f and when $f\in\mathfrak{R}_{\mathfrak{G}}{}^0$, one calls g a partial fraction representation of f. The author shows that $\mathfrak{R}_{\mathfrak{G}}=\mathfrak{R}_{\mathfrak{G}}{}^0\oplus A$ (where A is the linear space of functions analytic on X), that $\mathfrak{R}_{\mathfrak{G}_0}=\bigcap\mathfrak{R}_{\mathfrak{G}}$ and that $\mathfrak{R}_{\mathfrak{G}_0}{}^0=\bigcap\mathfrak{R}_{\mathfrak{G}}{}^0$, where $\mathfrak{G}_0$ is the class of all normal domains.

If f is meromorphic on the parameter disk U with a single pole of order n at $p\in U$, then $L^*_{X-U}f=\varphi_{p,n}$ is analytic on $X-U$ and $f-\varphi_{p,n}$ is analytic on U. Consequently, $\varphi_{p,n}$ is meromorphic on X with a single pole at p of order n. If prescribed locations and orders $\{(p,n)\}$ are given, then suitable coefficients may be found so that $\sum\varphi_{p,n}$ is compactly convergent on $X-(\bigcup p)$ to a function φ. This standard device is repeated by the author to obtain the additional information that $\varphi\in\mathfrak{R}_{\mathfrak{G}}{}^0$ for some $\mathfrak{G}$ and consequently $\varphi\in\mathfrak{R}_{\mathfrak{G}_0}{}^0$. A theorem of Florack [Schr. Math. Inst. Univ. Münster No. 1 (1948); MR **12**, 251] is then used to show that with poles prescribed one can even find a non-zero $\varphi\in\mathfrak{R}_{\mathfrak{G}_0}$; whether this φ belongs to $\mathfrak{R}_{\mathfrak{G}_0}{}^0$ remains open. Thus every meromorphic function is the product of an analytic function with a member of $\mathfrak{R}_{\mathfrak{G}_0}$.

Tillman's duality lemma [J. Reine Angew. Math. **195** (1956), 76–110; MR **17**, 510] is specialized (a net of functions analytic on G is compactly convergent if and only if the net $\int_{\partial G}f\,d\varphi$ converges for every differential $d\varphi$ drawn from a specified class), is reproved, and used to give a criterion for membership in $\mathfrak{R}_{\mathfrak{G}}$ and $\mathfrak{R}_{\mathfrak{G}}{}^0$. When X is the plane, Tillman's differentials can be exhibited and one has then a characterization of those meromorphic functions on the plane which are represented by the (infinite) series of their principal parts. *W. C. Fox* (New Orleans, La.)

12215:

Royden, H. L. The equation $\Delta u=Pu$, and the classification of open Riemann surfaces. Ann. Acad. Sci. Fenn. Ser. A I No. 271 (1959), 27 pp.

Let $c\,dx\,dy$ be the representative in an arbitrary local coordinate of a second order differential form P on the open Riemann surface W, such that c is non-negative and C^∞. The author is concerned with real functions u continuous on the closure of (usually, but not always) compact regions Ω (i.e., domains in W with compact closure and a boundary $\partial\Omega$ consisting of a finite number of disjoint analytic curves) such that u is C^∞ on Ω and such that $\Delta u=cu$ is satisfied for every local coordinate in Ω. Such u's are called solutions of $\Delta u=Pu$, $(*, P)$, on Ω. Myrberg [Math. Scand. **2** (1954), 142–152; Ann. Acad. Sci. Fenn. Ser. A I No. 170 (1954); MR **16**, 33, 34] has already shown that the maximum principle holds for such u, that every continuous function f on $\partial\Omega$ determines a solution of $(*, P)$ which agrees with f on $\partial\Omega$, that locally uniform limits of sequences of solutions (on W or on an exhaustion) of $(*, P)$ are solutions of $(*, P)$ on W, that uniform boundedness implies equicontinuity of a sequence of solutions of $(*, P)$ (so normal-family theory works) and that Harnack's theorem holds for the solutions of $(*, P)$. If $f=1$ on $\partial\Omega$, then the corresponding solution of $(*, P)$ on Ω is called the elliptic measure of Ω and is denoted by $w(\Omega)$. Since $0<w(\Omega)<1$, the sequence of elliptic measures of an exhaustion Ω_n of W converges (Harnack) to the identically zero, or else everywhere positive, elliptic measure $w(W)$ of W. (The pair $\{W, P\}$ is respectively called parabolic or hyperbolic ($P\not\equiv 0$). The largest solution of $(*, P)$ bounded by one is $w(W)$.) With the boundary value one applied to $\partial\Omega$, where $\Omega\subset\Omega_n$ for all n, and the value zero applied to $\partial\Omega_n$, the resulting sequence of solutions on $\Omega_n-\Omega$ converges to $w(\Omega, W-\Omega)$ and $w(W-\Omega)-w(\Omega, W-\Omega)=\lim w(\partial\Omega_n, \Omega_n-\Omega)$, is denoted by $w(\theta, W-\Omega)$, and is called the elliptic measure of the ideal boundary of W relative to $W-\Omega$. These measures are independent of the exhaustion and are used to prove that non-trivial bounded solutions of $(*, P)$ exist on W if and only if the relation $u\leqq\text{lub}\,\{u(x): x\in\partial G\}$ fails for some bounded solution u of $(*, P)$ on some domain G of non-compact closure. This result was given by Ozawa [Kōdai Math. Sem. Rep. **1952**, 63–76; MR **14**, 462] in a different formulation.

The differential forms under consideration are given a partial order: $Q\prec P$ means that for some compact region Ω and constant α in $(0, 1]$ one has $\alpha q\leqq p$ whenever q and p are coefficients of representatives in local coordinates in $W-\Omega$ of Q and P respectively. Thus if $Q\prec P$ and $\{W, Q\}$ is parabolic, then so is $\{W, P\}$. Moreover, there then exists an isometric isomorphism sending the Banach space BP of all bounded solutions of $(*, P)$ onto a subspace of BQ. If also $P\prec Q$, then this isometry is onto. The proof rests on the fact that BP is isomorphically isometric with the corresponding space of solutions on $W-\Omega$ which vanish on $\partial\Omega$, for any compact region Ω. With $P\equiv 0$, the equivalent facts (1) two independent bounded solutions of $(*, Q)$ exist on W and (2) some solution of $(*, Q)$ on W takes both positive and negative values, are seen to imply that a non-constant bounded harmonic function exists on W.

Let D denote the class of continuous real functions f on W which admit generalized total differentials, i.e., for which a square-integrable first-order differential df exists such that $\iint f\,d\eta+\iint\eta\,df=0$ for every smooth first-order differential, η, with compact carrier. One has a Dirichlet integral for functions in D, namely, $D(f, g)=\iint df*dg$. Two subspaces of D are distinguished: BD, the subspace of bounded functions and E, the subspace of all f for which $\iint f^2P<\infty$. When $P\not\equiv 0$, E has an inner product: $E(f, g)=D(f, g)+\iint fgP$. Thus those members f of BD which are limits of compactly convergent sequences $\{\varphi_n\}$ in K (K consists of all members of BD with compact carriers) for which $\{\varphi_n\}$ is uniformly bounded and $D(f-\varphi_n)\to 0$ are defined to be the members of $\bar{K}$. The members of $\bar{K}$ continue to "vanish on the ideal boundary" as far as integration-by-parts is concerned. Moreover, $\bar{K}$ is orthogonal to every solution of $(*, P)$ in BD, and to the harmonic functions with finite Dirichlet integrals. Every member f of BD is of the form $f=u+f_0=v+g_0$ where f_0 and g_0 are in $\bar{K}$, u is a solution $(*, P)$ in BD and v is harmonic. The orthogonal complement of K in E is the space of bounded solutions of $(*, P)$ with finite Dirichlet integral. Finally, $\bar{K}$ turns out to be closed under various weak kinds of convergence.

When attention is turned to the members of these spaces which are harmonic, one obtains various more precise results. The space of bounded harmonic functions (with finite Dirichlet integral) is isometrically isomorphic (isomorphic) with the space of bounded solutions of $(*, P)$ (with finite Dirichlet integral). Finally, the latter is dense in the space of all solutions of $(*, P)$ with finite

E-norm, and the former is dense in the space of harmonic functions with finite Dirichlet-norm. The paper concludes with some fragmentary results about unbounded solutions of (*, P) with finite Dirichlet integral.

W. C. Fox (New Orleans, La.)

12216:

Nakai, Mitsuru. The space of bounded solutions of the equation $\Delta u = pu$ on a Riemann surface. Proc. Japan Acad. **36** (1960), 267–272.

Let p be a smooth non-negative density on a Riemann surface R, and consider the equation $\Delta u = pu$. Let $B_p(R)$ denote the Banach space of all bounded solutions of this equation. The reviewer [#12215] has shown that if, for some constant a, $a^{-1}p \leq q \leq ap$ except on a compact subset of R, then B_p and B_q are isomorphic. The author gives the following different criterion for the isomorphism of B_p and B_q. Theorem: If two densities p and q or R satisfy the condition

$$\iint_R |p(z)-q(z)|\,dxdy < \infty,$$

then the Banach spaces $B_p(R)$ and $B_q(R)$ are isomorphic. Applications are made to the removability of sets with respect to this equation. *H. L. Royden* (Stanford, Calif.)

12217:

Tsuji, Masatsugu. Behnke-Stein's theorem on the existence of a basic differential on a compact Riemann surface with boundary curves. Comment. Math. Univ. St. Paul. **8** (1960), 45–51.

$\mathfrak{R}$ sei eine berandete kompakte Riemannsche Fläche, deren Rand aus endlich vielen analytischen Kurven besteht. Der Verfasser gibt einen relativ einfachen Beweis für die Existenz (1) eines Elementardifferentials, (2) von Differentialen erster Gattung mit gegebenen Perioden und (3) von meromorphen Funktionen mit gegebenen Polstellen auf $\mathfrak{R}$. Er benutzt, daß sich $\mathfrak{R}$ als Quotient des Einheitskreises nach einer Fuchs'schen Automorphismengruppe beschreiben läßt und konstruiert die den Differentialen und meromorphen Funktionen auf $\mathfrak{R}$ entsprechenden Funktionen im Einheitskreis.

K.-H. Spallek (Münster)

12218:

Matsumoto, Kikuji. An extension of a theorem of Mori. Japan J. Math. **29** (1959), 57–59.

Mori [Kōdai Math. Sem. Rep. **1952**, 57–58; MR **14**, 264] introduced the concept of a domain G in a Riemann surface R belonging to class SO_X (relative to R) provided every member of class X on G, continuous on G^- and zero on ∂G, vanishes on G. In particular, X may be HB or HD. Mori [J. Fac. Sci. Univ. Tokyo Sect. I **6** (1951), 247–257; MR **13**, 735] proved that if a surface R is of class O_{HB} [O_{HD}, respectively] then at least one of each disjoint pair of subdomains must belong to class $\mathrm{SO}_{\mathrm{HB}}$ [$\mathrm{SO}_{\mathrm{HD}}$, respectively] relative to R. According to the reviewer of Matsumoto [Nagoya Math. J. **15** (1959), 261–274; MR **21** #6429], the latter gave alternative proofs of results of Muramochi [Osaka Math. J. **7** (1955), 109–127; **10** (1958), 83–102; MR **17**, 26; **20** #3272] including a generalization of Mori's theorem (in which "two" is replaced by n)—which involves the classes $\mathrm{O}_{\mathrm{HB}_n}$ [$\mathrm{O}_{\mathrm{HD}_n}$, respectively] of surfaces whose ideal boundary consists of at most n HB [maximal HD, respectively]—indivisible sets. In the paper under review the author proves the converse to his generalization of Mori's theorem. Thus, e.g., a surface R is of class $\mathrm{O}_{\mathrm{HB}_n}$ if and only if at least one of every $n+1$ pairwise disjoint subsurfaces is necessarily in class $\mathrm{SO}_{\mathrm{HB}}$. The corresponding O_{HD} condition is more complicated.

W. C. Fox (New Orleans, La.)

12219:

Constantinescu, C. Über die Klassifikation der Riemannschen Flächen. Acta Math. **102** (1959), 47–78.

This is a continuation of a previous work by the author and A. Cornea [Nagoya Math. J. **13** (1958), 169–233; MR **20** #3273]. Let R be a Riemann surface of positive boundary and $\mathfrak{S}(R)$ be a family of quasi-bounded harmonic functions. The author defines the following concepts in terms of the universal covering surface of R: ideal boundary as a set, $\mathfrak{S}$-indivisibility of its subset, etc. He then obtains several results, among which the following is typical. Let R' be an end of R and M' be an $\mathfrak{S}$-indivisible set of R'. If each $u \in \mathfrak{S}(R)$ belongs to $\mathfrak{S}(R')$ as a function on R', then M' is an $\mathfrak{S}$-indivisible set on the ideal boundary of R. It is also shown that its converse holds under some restrictions. *H. Mizumoto* (Tokyo)

12220:

Jurchescu, Martin. Ensembles AB-enlevables dans des espaces complexes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 13–26. (Romanian. Russian and French summaries)

This paper discusses relations between several properties possible for subsets of a complex space. These properties include: nowhere disconnecting, analytically or plurisubharmonically imbedded, AB-removability, and dimension less by at least two than that of the space.

M. M. Day (Urbana, Ill.)

12221:

Jurchescu, Martin. On AB-removability in complex spaces. Rev. Math. Pures Appl. **4** (1959), 357–368.

AB-removable boundaries and point sets have been studied on Riemann surfaces [L. Sario, 11te Skandinav. Matematikerkongress (Trondheim, 1949), pp. 229–238, Johan Grundt Tanums Forlags, Oslo, 1952; MR **14**, 863; L. Ahlfors and A. Beurling, Acta Math. **82** (1950), 101–129; MR **12**, 171]. The author considers, more generally, AB-removable sets in a complex space of arbitrary dimension. A closed subset B of a complex α-space R is said to be AB-removable in R if, for any region U in R and for any component U_1 of $U-B$, every bounded holomorphic function on U_1 has a holomorphic continuation on all of U.

It is shown that an AB-removable subset of R is nowhere disconnecting in R. A sufficient condition is given for B to be AB-removable; the condition is satisfied by more general sets than analytically thin ones. As a counterpart of sets of logarithmic capacity zero on Riemann surfaces the author introduces pluriharmonically thin sets B in a complex α-space R. By definition, B has this property if, for each point $x \in B$, there exists a neighborhood U of x and a function $u \not\equiv -\infty$, plurisubharmonic in U, such that $B \cap U$ is contained in the set $\{x \in U; u(x) = -\infty\}$. It is demonstrated that a pluriharmonically thin set in R is AB-removable in R. As a

consequence the author obtains a generalization of the theorem of Radó-Behnke-Stein-Cartan [cf. E. Heinz, Math. Ann. **131** (1956), 258–259; MR **18**, 149].

L. Sario (Los Angeles, Calif.)

12222:

Maak, Wilhelm. Fastautomorphe Funktionen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 289–319 (1960).

In this paper the author discusses almost-periodicity of meromorphic functions on the upper half-plane $H=\{\tau=x+iy|y>0\}$ relative to the modular group G. The space of functions considered, called functions of meromorphic behaviour, is defined to consist of those meromorphic functions $f(\tau)$ in H such that: (1) $f(\tau)$ has only poles in H, in at most finitely many equivalence classes under the action of G; (2) $f(\tau)$ has bounded order of growth upon approaching the poles and the cusps of G (the rational boundary points of H); (3) for any closed subset $D\subset H$ not containing poles of $f(\tau)$, $\|f\|_D=\sup_{T\in G,\,\tau\in D}|f(T\tau)|<\infty$. It then follows that for an open subset $U\subset D$ this space of functions is a Banach space with the norm $\|f\|_U$ as introduced above; and letting G act on this space by putting $Tf(\tau)=f(T\tau)$, $T\in G$, it follows that $\|Tf\|_U=\|f\|_U$. A function $f(\tau)$ as above is called almost-automorphic at a point τ_0 if for some open neighbourhood U of τ_0 the map $T\to Tf$ is an almost-periodic function from G into the function space with norm $\|f\|_U$. The author shows a certain independence from the point τ_0, except of course that the poles of $f(\tau)$ must be omitted. Also the restrictions of $f(\tau)$ to lines $\operatorname{Im}\tau=\eta$, for η suitably large, are almost-periodic on those lines in the customary sense, with customary Fourier expansions. The general results from the Bohr theory apply, so that the sum, convolution, uniform limit, and mean values of almost-periodic functions are again almost-periodic. The concepts are such that almost-periodic functions without poles must be constants; so that the admission of poles is a natural feature. The featured result of the paper is that all finite-dimensional bounded representations of the modular group can be realized by such almost-automorphic functions.

R. C. Gunning (Princeton, N.J.)

12223:

Kaufhold, Günter. Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades. Math. Ann. **137** (1959), 454–476.

Let $Z=X+iY$ be an $n\times n$ complex symmetric matrix with $Y>0$. Let A, B, C, D be $n\times n$ matrices with rational integral entries, satisfying certain conditions, and let $\{A, B\}$ be the set of pairs UA, UB with U unimodular. It is known that the Dirichlet series

$$\varphi(s)=|Y|^{s/2}\sum_{\{A,B\}}\operatorname{abs}(AZ+B)^{-s},$$

$$\psi(s)=|Y|^{s/2}\sum_{\{C,D\}}\operatorname{abs}(CZ+D)^{-s}$$

have $n+1$ as the exact abscissa of absolute convergence [H. Braun, Math. Z. **44** (1939), 387–397]. (Here $|A|$ is the determinant and abs A is the absolute value of the determinant of the matrix A.) The author inquires as to the analytic continuation of $\varphi(s)$, $\psi(s)$ and the existence of functional equations which they satisfy.

He confines himself to the case $n=2$ and proves: $\varphi(s)$ is meromorphic in the whole s-plane and has a pole of first order at $s=3$ with residue $90/\pi^2$. It satisfies the functional equation $\omega(s)=\omega(3-s)$, where $\omega(s)=\rho(s)\rho(2s-2)\varphi(s)$ and $\rho(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Similar results hold for $\psi(s)$. Some of the intermediate calculations are carried out for $n>2$.

J. Lehner (E. Lansing, Mich.)

12224:

Rodin, Yu. L. The characteristic functions of certain integral equations. Dokl. Akad. Nauk SSSR **130** (1960), 23–25 (Russian); translated as Soviet Math. Dokl. **1**, 13–15.

R sei eine kompakte Riemannsche Fläche und $A(\zeta, z)d\zeta$ ein Elementardifferential (Analogon zum Cauchy-Kern) auf R: es ist in ζ ein Differential 3. Gattung mit den einfachen Polen z und P_0 sowie den Residuen $+1$ bzw. -1, in z ist es eine meromorphe Funktion mit den Polen ζ und P_μ $(\mu=1,\cdots,p)$; dabei sind die P_μ $(\mu=0, 1,\cdots,p)$ geeignete feste paarweise verschiedene Stellen auf R. T sei ein Gebiet von R, das keines der P_μ $(\mu=0, 1,\cdots,p)$ enthält und $\overline{B(z)}dz$ ein dort holomorphes Differential. Die Gleichung $U_z=B(z)U$ besitzt eine Lösung, die auf R meromorph fortsetzbar ist, deren Pole die P_μ $(\mu=1,\cdots,p)$ sind und die in P_0 verschwindet. Diese Funktion löst gleichzeitig die Integralgleichung

$$U(z)+\pi^{-1}\iint_T B(f)\overline{U(f)}A(f,z)\,dT=0.$$

H. Tietz (Münster)

12225:

Bagemihl, F.; Seidel, W. Behavior of meromorphic functions on boundary paths, with applications to normal functions. Arch. Math. **11** (1960), 263–269.

Results concerning the boundary properties of a function $f(z)$, meromorphic in the unit disk, are derived as a consequence of the assumed behaviour of $f(z)$ on a boundary path Λ whose end contains more than one point. In addition to theorems on the behaviour of a holomorphic function $f(z)$ in certain disks, which are the analogues in the unit disk of the "Milloux cercles de remplissage" for entire functions, new information is obtained about the boundary properties of normal meromorphic functions. [See the reviewer and K. I. Virtanen, Acta Math. **97** (1957), 47–65; MR **19**, 403 and #12226, by the above authors.] Results of this kind which can easily be restated here are that if a normal function tends to a limit along a boundary path whose end contains more than one point, then the function is identically constant, or that every normal holomorphic function has a Fatou point.

O. Lehto (Helsinki)

12226:

Bagemihl, F.; Seidel, W. Sequential and continuous limits of meromorphic functions. Ann. Acad. Sci. Fenn. Ser. A I No. 280 (1960), 17 pp.

Let C denote the unit circle and D the open unit disk. Let $f(z)$ be meromorphic in D, and $\{z_n\}$ be a sequence of points in D that converges to a point $\zeta\in C$. Assuming that $\lim f(z_n)=c$, the authors study under what conditions on $f(z)$ or on the sequence of points can it be inferred that $f(z)\to c$, as $z\to\zeta$ in some continuous manner.

If the function $f(z)$ is normal in D, i.e., if the family

$\{f(S(z))\}$ is normal, where $z'=S(z)$ is an arbitrary one-to-one conformal mapping of D onto itself, then the following results are shown to hold. If $f(z)\neq c$ in D and if the non-Euclidean distance ρ_n of z_n and z_{n+1} is bounded, then $f(z)$ has the angular limit c at ζ. If $\rho_n\to 0$, the requirement $f(z)\neq c$ can be dropped. Examples show that neither the condition of normalcy nor the conditions on the point sequence can be dropped.

The paper is concluded by some identity and uniqueness theorems. If $f(z)$ is of bounded characteristic in D, and if the sequence $\{z_n\}$ has at least two limit points on C such that $|z_n|\to 1$, $\rho_n=O(1)$, then $\lim f(z_n)=c$ implies $f(z)\equiv c$. The result remains valid if $f(z)$, instead of being of bounded characteristic, omits at least three values in D. A counterexample shows that the theorem does not hold for arbitrary normal functions. *O. Lehto* (Helsinki)

12227:

Jenkins, James A. Some problems for typically real functions. Canad. J. Math. **13** (1961), 299–304.

For a typically real function

$$f(z) = z+a_2z^2+\cdots+a_nz^n+\cdots, \qquad |z| < 1,$$

sharp upper and lower bounds for $f(r)$ and $f'(r)$ in terms of r and a_2 are obtained when r is real. For $0<r<1$,

$$r(1-a_2r+r^2)^{-1} \leq f(r) \leq \tfrac{1}{4}(2+a_2)r(1-r)^{-2} + \tfrac{1}{4}(2-a_2)r(1+r)^{-2},$$

$$(1-r^2)(1-a_2r+r^2)^{-2} \leq f'(r) \leq \tfrac{1}{4}(2+a_2)(1+r)(1-r)^{-3} + \tfrac{1}{4}(2-a_2)(1-r)(1+r)^{-3}.$$

The upper bounds are attained only by the function

$$\tfrac{1}{4}(2+a_2)z(1-z)^{-2}+\tfrac{1}{4}(2-a_2)z(1+z)^{-2},$$

and the lower bounds only by the function $z(1-a_2z+z^2)^{-1}$. Analogous bounds for negative r hold since $-f(-z)$ is also typically real.

The proof depends upon the reviewer's Stieltjes integral representation for $f(z)$ and the Neyman-Pearson Lemma (the author gives a proof of the special case needed). To take care of the maximum problem the author proves the following lemma.

Let $g(x)$, $h(x)$ be non-negative measurable functions defined on the interval $[a, b]$ such that $g(x)/h(x)$ is continuous and strictly increasing. Let $0\leq k\leq \int_a^b g(x)dx$. Then the extremal problem given by $\int_a^b f(x)g(x)dx=k$, $\int_a^b f(x)h(x)dx=$ maximum for functions $f(x)$ increasing on $[a, b]$ and satisfying $0\leq f(x)\leq 1$ has a solution $f^*(x)$ defined by $f^*(x)=k/\int_a^b g(x)dx$, $a\leq x\leq b$. This solution is uniquely determined apart from its values at a and b.

M. S. Robertson (New Brunswick, N.J.)

12228:

Reich, Edgar; Warschawski, S. E. Canonical conformal maps onto a circular slit annulus. Scripta Math. **25** (1960), 137–146.

In an earlier paper [Pacific J. Math. **10** (1960), 965–985; MR **22** #8120] the authors showed that the circular slit-disk mapping can be obtained by an extremal method. In the present paper they deal with the mapping onto a circular slit annulus by a similar method. Let Ω be a domain on the z-plane which separates 0 and ∞, both being supposed to be exterior points. Let $\mathfrak{F}=\mathfrak{F}(\Omega)$ denote the family of analytic functions $f(z)$ with $|f(z)|<1$ and univalent in Ω such that $w=f(z)$ maps every closed curve in Ω with index $+1$ with respect to $z=0$ into a curve having index $+1$ with respect to $w=0$. Let ρ be defined by $\rho_f=\sup_{z\in\Omega}|f(z)|$, $\rho=\sup_{f\in\mathfrak{F}}\rho_f$. Then it is shown that there exists a function $\phi(z)\in\mathfrak{F}$ with $\rho=\rho_\phi$, and $w=\phi(z)$ maps Ω onto the annulus $\rho<|w|<1$ slit along concentric circular arcs which may possibly reduce to single points.

Let $\{\Omega_n\}$ be an exhaustion of Ω such that each Ω_n separates 0 from ∞ and is bounded by n closed Jordan curves. Then the corresponding extremal function $\phi_n(z)$ tends, as $n\to\infty$, to $\phi(z)$ uniformly in any compact subset in Ω, if $\phi_n(z)$ and $\phi(z)$ are normalized by $\arg\phi_n(z_0)=\arg\phi(z_0)=0$ at an interior point $z_0\in\bigcap_{\nu=0}^{\infty}\Omega_\nu\cap\Omega$. This implies that the extremal function $\phi(z)$ for Ω is essentially unique.

Finally, it is shown that the mapping onto a circular slit disk can be obtained as a limiting case of that onto a circular slit annulus. *Y. Komatu* (Tokyo)

12229:

Merkes, E. P.; Scott, W. T. On univalence of a continued fraction. Pacific J. Math. **10** (1960), 1361–1369.

The authors employ methods originally developed by Thale [Proc. Amer. Math. Soc. **7** (1956), 232–244; MR **17**, 1063] to obtain the radius of univalence $U(\alpha)$, as well as bounds for the starlike radius and the radius of convexity for the family of functions $K(\alpha)$. A function belongs to $K(\alpha)$ (α a positive integer) if it has a C-fraction expansion of the form $z/(1+K(a_nz^n))$, where $|a_n|\leq\frac{1}{4}$ for all $n=1, 2, \cdots$. It is shown that $U(2)=(2\sqrt{2})/3$ and that for $\alpha\neq 2$

$$(U(\alpha))^\alpha = \frac{6\sqrt{(\alpha^2-2\alpha+9)}-2(\alpha+7)}{(\alpha-2)^2}.$$

W. J. Thron (Boulder, Colo.)

12230:

Хейман, В. К. [Hayman, W. K.]. ★Многолистные функции [Multivalent functions]. Translated from the English by V. P. Havin; edited by I. E. Bazilevič. Izdat. Inostr. Lit., Moscow, 1960. 180 pp. 5.90 r.

A translation into Russian of W. K. Hayman's *Multivalent functions* [Cambridge Univ. Press, New York, 1958; MR **21** #7302]. The translation has been generously amplified with comments and annotations of the editor.

A. J. Lohwater (Providence, R.I.)

12231:

Strelicas, Š. Some properties of functions analytic in a circle. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **4** (1955), 67–71. (Lithuanian. Russian summary)

Author's summary: "The following propositions are proved. (1) Let $w=f(z)=\sum_1^\infty a_nz^n$ be a function analytic in $|z|<R$. The following estimate holds:

$$|a_n| \leq \frac{2n}{\rho^n}\int_0^\rho \frac{K_\vartheta(r)}{r}\,dr \qquad (n = 1, 2, \cdots; \rho < R),$$

where $K_\vartheta(r)=\sup_{|z|=r}|f(z)|$ for $\arg f(z)=\vartheta$, $0\leq\vartheta<2\pi$. The estimate is exact in the sense that there exist functions for which equality holds. (2) Let $w=f(z)$, $f(0)=0$, be a function analytic in $|z|\leq r$. If there is a ϑ, $0\leq\vartheta<2\pi$, such that the function $w^*=e^{-i\vartheta}zf'(z)$ assumes on the circle $|z|=r$ only one positive and one negative value, each at only one point, then the function $w=f(z)$ is schlicht for $|z|\leq r$."

12232:

Itô, Jun-iti. The coefficient problem of regular functions. Proc. Amer. Math. Soc. **12** (1961), 53–60.

Continuing his studies of meromorphic functions $f(z)$ in the unit circle for which $\Re f(z)$ changes sign on $|z|=1$ a finite number of times [Trans. Amer. Math. Soc. **89** (1958), 60–78; MR **20** #2455] the author obtains six theorems concerned with bounds on the coefficients under weaker conditions than those considered in his earlier paper. We shall describe here two of the theorems.

Let $B(S, M)$ denote the class of functions $f(z)$, regular in $0 \leq \rho \leq |z| < 1$ for which

$$(1) \qquad \int_{-\pi}^{\pi} |\Re T(re^{i\phi})|\, d\phi < c < \infty \qquad (\rho < r < 1),$$

where $T(z) = f(z)g(s, z)$, $g(s, z) = Kz^{-s}\prod_{k=1}^{2s}(e^{i\sigma_k} - z)$, $|K| = 1$, $-\pi \leq \sigma_k \leq \pi$, and

$$(2) \qquad \lim_{r\to 1} \int_{-\pi}^{\pi} [\Re T(re^{i\phi})]^{-}\, d\phi \leq 2\pi M \qquad (M \geq 0),$$

where $[\Re T(z)]^{-} = \operatorname{Max}\{0, -\Re T(z)\}$. Let $B'(S, M)$ be the subclass of $B(S, M)$ for which $0, \pi$ are members of the set σ_k and for which $e^{-i\sigma_k}$ is a zero of $g(s, z)$ of the same order as the order of $e^{i\sigma_k}$.

The author's first theorem is that if $f(z) = \sum_{-\infty}^{\infty} a_n z^n \subset B(S, M)$, then, for all $n > s$,

$$(A) \qquad \begin{aligned} |a_n + \delta\bar{a}_{-n}| \leq{}& \sum_{k=0}^{s-1} \Delta(s, k, n)|a_k + \delta\bar{a}_{-k}| \\ &+ \Delta(s, s, n)(|a_s + \delta(\bar{a}_{-s} + K^2M)| + M), \end{aligned}$$

where $\delta = K^2 \exp(-i\sum_1^{2s}\sigma_k)$, with equality for $M = 0$, $\sigma_k = 0$, $k = 1, 2, \cdots, 2s$. The inequality (A) for the case $\delta = 1$, $M = 0$ was obtained by the reviewer, and, for the notation, see [Canad. J. Math. **4** (1952), 407–423; MR **14**, 460].

The author's second theorem is that if $f(z) = \sum_1^{\infty} a_n z^n \subset B'(S, M)$, then

$$(B) \qquad |a_n| \leq \sum_{k=1}^{s} D(s, k, n)|a_k| + 2D(s, s, n)M,$$

with equality if $M = 0$. The inequalities (B) were obtained when $M = 0$ for the class $T(S)$ of typically-real functions of order S by A. W. Goodman and M. S. Robertson [Trans. Amer. Math. Soc. **70** (1951), 127–136; MR **12**, 691].

M. S. Robertson (New Brunswick, N.J.)

12233:

Lai, Wan-tzei. Über die Konjektur von Goodman für die beinahe beschränkten Funktionen. Acta Math. Sinica **9** (1959), 292–294. (Chinese. German summary)

Let $f(z) = a_1 z + a_2 z^2 + \cdots$ be regular in the unit circle and almost bounded with respect to a group that satisfies certain additional conditions set forth by the reviewer [Trans. Amer. Math. Soc. **78** (1955), 82–97; MR **16**, 685]. It was proved by the reviewer that if $f(z)$ is univalent, then $|a_1| \leq 1$, with equality only if $f(z) = \eta z$, $|\eta| = 1$. The author shows that the condition of univalence can be dropped and the same conclusion still holds. {Because of the language barrier, the reviewer is unable to pass on the correctness of the proof. He hopes that the author will soon publish his proof in a Western language.}

A. W. Goodman (Lexington, Ky.)

12234:

Akutowicz, Edwin J. Sur l'approximation par certaines fonctions entières. C. R. Acad. Sci. Paris **250** (1960), 1955–1957.

Summary of #12235. *P. Koosis* (Paris)

12235:

Akutowicz, Edwin J. Sur l'approximation par certaines fonctions entières. Ann. Sci. École Norm. Sup. (3) **77** (1960), 281–301.

Pour $a > 0$, soit E_a l'ensemble des fonctions φ de la forme $\varphi(t) = \int_{-b}^{b} e^{it\omega} d\mu(\lambda)$, μ étant une mesure finie quelconque et b un nombre positif quelconque $< a$. On note par H un espace de l'un des deux types suivants:

$$H = L_p(d\omega)$$

$$(1) \qquad = \left\{ f \;\middle|\; \|f\|_p = \left[\int_{-\infty}^{\infty} |f(t)|^p\, d\omega(t)\right]^{1/p} < \infty \right\} \qquad (1 \leq p < \infty),$$

ω étant une mesure positive finie. (Evidemment, $E_a \subset L_p(d\omega)$.)

$$(2) \qquad H = \left\{ f \text{ continue} \;\middle|\; \lim_{|t|\to\infty} \frac{f(t)}{\Omega(t)} = 0 \right\},$$

$\Omega(t)$ étant une fonction positive continue, et telle que $\lim_{|t|\to\infty} \varphi(t)/\Omega(t) = 0$ pour toute $\varphi \in E_a$. Ici, on pose $\|f\| = \sup_t |f(t)/\Omega(t)|$ pour $f \in H$. Dans la suite on notera toujours par $\|\ \|$ la norme de l'espace H.

L'auteur étudie le problème suivant: Sous quelles conditions (portant sur a et l'espace H) est E_a dense dans H?

L'auteur obtient plusieurs conditions qui, bien que ne mettant pas en évidence une dépendance explicite sur la valeur de a, sont toutes nécessaires et suffisantes.

Dans deux de ses réponses intervient une fonction $b(z)$ définie de la manière suivante: soit

$$A = \left\{ \varphi \in E_a \;\middle|\; \left\| \frac{\varphi(t)}{(1+|t|)} \right\| \leq 1 \right\}.$$

Pour chaque z complexe on pose $b(z) = \sup_{\varphi\in A} |\varphi(z)|$. (Notons que $1 \in E_a$, donc pour un $c > 0$ on aura $c \in A$, c'est-à-dire, $b(z) \geq c$ pour tout z. Pour la commodité des démonstrations, l'auteur suppose la norme $\|\ \|$ choisie de telle manière qu'on puisse prendre $c = 1$, ce qui ne restreint nullement la généralité des résultats.)

En lisant l'article, il est plus commode de passer directement du § 1 au § 3. Là, on démontre: $\overline{E_a} = H$ si et seulement si $b(z) = \infty$ pour au moins un z non réel.

La démonstration, assez simple, est effectuée en examinant l'appartenance ou non-appartenance de fonctions de la forme $1/(t-c)$ ($\operatorname{Im} c \neq 0$) à $\overline{E_a}$, et repose sur le fait que $\varphi \in E_a$ entraîne $(\varphi(t) - \varphi(c))/(t-c) \in E_a$ pour chaque c complexe.

Un autre résultat est celui-ci: $\overline{E_a} = H$ si et seulement si $\int_{-\infty}^{\infty} \log b(t)/(1+t^2)\, dt = \infty$. La nécessité est une conséquence presque immédiate du théorème ci-dessus. Quant à la suffisance, c'est démontré dans le § 2, de la manière suivante: Si $\overline{E_a} \neq H$, il existe une fonctionelle linéaire $\Theta \neq 0$ sur H telle que $\Theta(E_a) = 0$. Supposons que $H = L_p(d\omega)$. Il y a une fonction $\vartheta \in L_1(d\omega)$ telle que $\Theta(f) = \int_{-\infty}^{\infty} f(t)\vartheta(t)\, d\omega(t)$, $f \in H$. Pour $\operatorname{Im} z > 0$, soit $B_\vartheta(z) = \int_{-\infty}^{\infty} \vartheta(t)/(t-z)\, d\omega(t)$. Par un raisonnement dont

la forme revient à Carleman, l'auteur montre que $\Theta(E_a)=0$ entraîne la relation

$$(*) \qquad \Theta\left(\frac{\varphi(t)}{t-z}\right) = \varphi(z)B_\Theta(z),$$

valide pour $\varphi \in E_a$ et $\operatorname{Im} z>0$. De cette formule on déduit assez facilement une majoration pour $\log|\varphi(z)|$ qui permet de conclure

$$\int_{-\infty}^{\infty} \frac{\log b(t)}{1+t^2}\,dt < \infty.$$

Pour les espaces H de type 2, la démonstration se fait d'une manière analogue.

Toujours supposant que $\overline{E_a} \neq H$, l'auteur étudie dans le § 4 la fonction

$$B_\Theta(z) = \sup\{|B_\Theta(z)| \;\big|\; \Theta(E_a)=0,\ \|\Theta\| \leq 1\},$$

définie pour $\operatorname{Im} z>0$. Il montre que $B_\Theta(z)>0$ pour $\operatorname{Im} z>0$, ce qui, avec la relation (*), fournit une autre démonstration du fait que $b(z)<\infty$ pour chaque z non réel si $\overline{E_a}\neq H$. Pour les espaces $H=L_p(d\omega)$ avec $1<p<\infty$ il est démontré que

$$B_\Theta(z) = \inf_{\varphi\in E_a} \left\| \frac{1}{t-z} - \varphi(t) \right\|, \quad \operatorname{Im} z > 0.$$

Le résultat du § 5 est que, si $H=L_p(d\omega)$ avec $1<p<\infty$, $\overline{E_a}=H$ si et seulement si

$$\inf\left\{ \left\| \frac{\varphi(t)}{(t-z)} \right\| \;\middle|\; \varphi\in E_a, \varphi(z)=1 \right\} = 0$$

pour un z non réel.

Partant de la formule (*), l'auteur démontre dans le § 6, que: Si $\overline{E_a}\neq H$, chaque $f\in\overline{E_a}$ est (p.p. $(d\omega)$ si $H=L_p(d\omega)$) la restriction à l'axe réel d'une fonction entière de type exponentiel, ayant un diagramme indicateur $\subset[-ia, ia]$.

L'étude s'achève avec une application au problème de l'extrapolation de fonctions de type positif, définies sur un segment (§ 7).

{Nota (quelques petites rectifications). P. 286, l. 9 à.p. du bas: Remplacer l'inégalité double

$$|\varphi(z)B_\Theta(z)| \leq A(z,\mathfrak{A}) \leq A(i,\mathfrak{A}) < \infty$$

par

$$|\varphi(z)B_\Theta(z)| \leq A(z,\mathfrak{A}) \leq A(1+|x|).$$

Car si $\|\varphi(t)/(1+|t|)\|\leq 1$ et $y\geq 1$, on sait seulement que $\|\varphi(t)/(t-z)\|\leq O(1+|x|)$ pour $z=x+iy$. Ensuite, définir la fonction $a(t)$ par la formule $a(t)=A(1+|t|)/|B_\Theta(t+i)|$ et la suite de la démonstration sera valable sans changement, puisque $\int_{-\infty}^{\infty}\log(1+|t|)/(1+t^2)dt<\infty$.

{P. 289, l. 3 à p. du bas: Ce qui est démontré, c'est que

$$\frac{1}{b(a_2)} \leq \frac{1}{|\varphi(a_1)|}\left(\frac{|a_2-a_1|}{\operatorname{Im} a_1}+1\right)+\frac{|a_2-a_1|}{\operatorname{Im} a_1}$$

et l'on aura une inégalité analogue avec $|\varphi(a_1)|$ remplacé par $b(a_1)$. Comme on peut supposer $b(a_1)\geq 1$ on aura tout de même l'inégalité sur la dernière ligne de la p. 289.

{P. 297: Dans l'énoncé du théorème 4, remplacer $(-ia, ia)$ par $[-ia, ia]$. (Autrement, le résultat cesse d'être vrai.)}

P. Koosis (Paris)

12236:

Hâstad, Matts. Uniform approximation with Diophantine side-conditions of continuous functions. Ark. Mat. **3** (1958), 487–493.

Let P be a set situated on a line segment, the extension of which does not intersect $I=[-1,1]$ perpendicularly, having a subset which is dense in itself. Let $\overline{P}$ be the image of P under reflection in the real axis. Each continuous function admits uniform approximation in I by the quotient of two polynomials, the zeros of the numerator belonging to I, P and $\overline{P}$, and those of the denominator to P and $\overline{P}$.

A polynomial whose coefficients are rational integers is called an I-polynomial. A necessary and sufficient condition that each continuous function in several real variables admit uniform approximation on a compact set A by an I-polynomial is the existence of an I-polynomial u so that $u\neq 0$ and $|u|<1$ holds in A. Let M denote the class of compact sets B for which there exist an I-polynomial $u(x,y)$ with $u\not\equiv 0$ and (*) $|u|<1$ in B. Let $N(B)$ be the subset of B such that $(\bar{x},\bar{y})\in N(B)$ implies $u(\bar{x},\bar{y})=0$ for every I-polynomial satisfying (*). If $B\in M$, those and only those continuous functions f admit uniform approximation by polynomials for which there exists an I-polynomial p such that $f(x,y)=p(x,y)$ for $(x,y)\in N(B)$. Finally, some special results are discussed.

K. Tandori (Szeged)

12237:

Stark, J. M. Minimum problems in the theory of pseudo-conformal transformations and their application to estimation of the curvature of the invariant metric. Pacific J. Math. **10** (1960), 1021–1038.

Soit B un domaine de C^2 possédant un noyau de Bergmann $K^{(B)}(z_1, z_2, \bar{t}_1, \bar{t}_2)$, et soit $K^{(B)}(z_1, z_2)=K^{(B)}(z_1, z_2, \bar{z}_1, \bar{z}_2)$; on désigne par $\lambda_B^{(1)}$ le minimum de $\int_B |f(z)|^2 d\omega$ parmi toutes les fonctions f, analytiques dans B, qui satisfont à un système (i) de conditions auxiliaires. Poursuivant une étude antérieure [même J. **6** (1956), 565–582; MR **18**, 474], l'auteur établit diverses relations entre les valeurs de $\lambda_B^{(1)}$ correspondant à divers systèmes (i). Il en déduit une majoration de la courbure riemannienne de la métrique invariante

$$ds^2 = \frac{\partial^2 \log K^{(B)}}{\partial z_m \partial \bar{z}_n} dz_m d\bar{z}_n \quad (m,n=1,2).$$

Cette majoration fait intervenir deux domaines de comparaison I, A, tels que $I\subset B\subset A$.

J. Lelong (Paris)

12238:

Wermer, John. An example concerning polynomial convexity. Math. Ann. **139**, 147–150 (1959).

Es wird eine Abbildung F des abgeschlossenen Dizylinders D in den 3-dimensionalen komplexen Zahlenraum angegeben und gezeigt, dass $F(D)$ mit seiner holomorphkonvexen Hülle nicht übereinstimmt.

H. Holmann (Münster)

12239:

Wermer, John. Addendum to "An example concerning polynomial convexity". Math. Ann. **140** (1960), 322–323.

Es wird ein Holomorphiegebiet im dreidimensionalen komplexen Zahlenraum konstruiert, das zum Polyzylinder

$|z_i| < 1$, $i = 1, 2, 3$, holomorph äquivalent ist, aber kein Rungesches Gebiet darstellt. *H. Holmann* (Münster)

12240:

Bavrin, I. I. Estimates of Taylor's coefficients for functions of many complex variables. Dokl. Akad. Nauk SSSR **131** (1960), 1231–1233 (Russian); translated as Soviet Math. Dokl. **1**, 398–400.

Let F be a regular function of the complex variables $z_1, \cdots, z_n$ in a multi-circular domain. In the case of two complex variables such domains were introduced by Temlyakov [same Dokl. **120** (1958), 976–979; MR **20** #5884]. A multi-circular domain D, which includes the hypersphere and hypercone as special cases, is a subdomain, containing the origin, of a certain domain of regularity bounded by a non-analytic $(2n-1)$-dimensional surface

$$\begin{aligned} |z_1| &= r_1(\tau_1)\cdots r_1(\tau_{n-1}), \\ |z_i| &= r_2(\tau_{i-1})r_1(\tau_i)\cdots r_1(\tau_{n-1}) \quad (2 \leqq i \leqq n-1), \\ |z_n| &= r_2(\tau_{n-1}), \\ 0 &\leqq \tau_i \leqq 1 \quad (1 \leqq i \leqq n-1), \end{aligned}$$

where $r_1(0)=0$, $r_1'(\tau)>0$ in $(0, 1]$, $r_1(1)<\infty$, $r_2(\tau) = \exp\left[-\int \tau/(1-\tau)\, d \ln r_1(\tau)\right]$ $(r_2(1)=0)$. Estimates are obtained for the coefficients in the Taylor series expansion of F when (i) F is bounded in D, (ii) $\operatorname{Re} F > 0$ on D. Examples of functions reaching these estimates are given. Proofs are indicated. *J. Mitchell* (University Park, Pa.)

12241:

Look, K. H. Schwarz lemma in the theory of functions of several complex variables. Acta Math. Sinica **7** (1957), 370–420. (Chinese. English summary)

Author's summary: "In this paper the author gives a complete and detailed proof of the results announced in Sci. Record (N.S.) **1** (1957), no. 2, 5–8 [MR **19**, 644]. Besides, there are generalizations and consequences of the previous results as follows. (1) If a bounded schlicht domain $\mathscr{D}$ is transitive, there exists a positive constant k depending only on $\mathscr{D}$ such that for any bounded schlicht domain $\mathscr{G}$ and any analytic mapping $w=f(z)$ carrying $\mathscr{G}$ into $\mathscr{D}$ we always have

$$\frac{\partial w}{\partial z}\, T_{\mathscr{D}}(w, \overline{w})\, \overline{\frac{\partial w'}{\partial z}} \leqq k^2 T_{\mathscr{G}}(z, \bar{z}).$$

(2) If $\mathscr{D}$ is a bounded schlicht transitive domain and if there is always a unique geodesic passing through any two points of $\mathscr{D}$, then, for any two points $z_1, z_2 \in \mathscr{D}$ and any inner analytic mapping carrying z_1, z to w_1, w_2 respectively, we always have $x(z_1, z_2) \leqq k_0(\mathscr{D}) x(w_1, w_2)$, where $x(z_1, z_2)$ is the geodesic distance between z_1 and z_2, and $k_0(\mathscr{D})$ the Schwarz constant of $\mathscr{D}$. (3) If $\mathscr{R}$ is the topological product of the classical domains $\mathscr{R}^{(1)}$, $\mathscr{R}^{(2)}$, $\cdots$, $\mathscr{R}^{(\mathcal{O})}$, then the Schwarz constant of $\mathscr{R}$ satisfies the relation

$$k_0(\mathscr{R}) = \sqrt{(k_0^2(\mathscr{R}^{(1)}) + k_0^2(\mathscr{R}^{(2)}) + \cdots + k^2(\mathscr{R}^{(\mathcal{O})}))}."$$

12242:

Tsuboi, Teruo. Bergman representative domains and minimal domains. Japan. J. Math. **29** (1959), 141–148.

Let $w(z)$ be a vector function defined on n-dimensional complex Euclidean space, $z \equiv (z_1, \cdots, z_n)$, that is, $w(z)$ is a row-vector with analytic elements $w_i(z)$ $(1 \leqq i \leqq n)$. Let D be a bounded domain. Using Bergman's definition of representative domain [*Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques*, Gauthier-Villars, Paris, 1947; MR **11**, 344], it is proved that a necessary and sufficient condition that D be a representative domain with center at a point u of D is that $T_D(z, \overline{u}) = T_D(u, \overline{u}) =$ constant matrix $(z \in D)$, where

$$T_D(z, \bar{t}) \equiv (\partial^2 \log K_D(z, \bar{t})/\partial z_i \partial \bar{t}_j) \quad (1 \leqq i, j \leqq n),$$

and $K_D(z, \bar{t})$ is the Bergman kernel function of D.

A minimal domain of D with respect to a fixed point u of D is defined as the domain with smallest Euclidean volume among all domains which are images of D by any pseudo-conformal vector function, satisfying certain normalizing conditions [loc. cit.]. A necessary and sufficient condition that the representative domain of D with respect to a fixed point u of D is also a minimal domain of D with respect to u is that

$$\begin{aligned} K_D(z, \overline{u})/\det T_D(z, \overline{u}) &= K_D(u, \overline{u})/\det T_D(u, \overline{u}) \\ &= \text{constant } (z \in D). \end{aligned}$$

The proof uses a similar theorem due to Maschler [Pacific J. Math. **6** (1956), 501–516; MR **18**, 473]. The results are extended to product domains.

J. Mitchell (University Park, Pa.)

12243:

Charzynski, Zygmunt. Bounds for analytic functions of two complex variables. Math. Z. **75** (1960/61), 29–35.

Let $\mathfrak{R}$ denote a bounded domain in the four-dimensional space of two complex variables z_1, z_2. The boundary of this domain consists of n uniformized analytic hypersurfaces i_k^3, $k=1, \cdots, n$, given by (1) $z_1^* = h_{1k}(z, \lambda)$, $z_2^* = h_{2k}(z, \lambda)$, $|z| \leqq 1$, $-\infty < \lambda < +\infty$. Let $f(z_1, z_2)$ be analytic in $\mathfrak{R}$. Let $f(h_{1k}, h_{2k}) \in \Psi$, where Ψ is a given normal family of functions $\psi(z)$ in $|z|<1$. Let $M(R) = \sup_{\psi \in \Psi} \sup_{|z| \leqq R < 1} |\psi(z)|$. Let $\mathfrak{B}_0$ denote the complement of $\mathfrak{R}$ with respect to the full space, $\mathfrak{B}_1$ the set of all points z_1, $z_2 \in \mathfrak{R}$ which belong to all of the analytic surfaces $z_1 = g_1(\zeta)$, $z_2 = g_2(\zeta)$, $|\zeta| \leqq 1$, lying in $\mathfrak{R} + i_{kR}{}^3 + \mathfrak{B}_0$ ($i_{kR}{}^3$ denotes the part of $i_k{}^3$ consisting of the points (1) corresponding to $|z| < R$), and having its boundary in $\mathfrak{B}_0$. Let $\mathfrak{B}_{m+1}$ denote the set of all points $z_1, z_2 \in \mathfrak{M} - (\mathfrak{B}_1 + \cdots + \mathfrak{B}_m)$ which belong to all of the analytic surfaces $z_1 = g_1(\zeta)$, $z_2 = g_2(\zeta)$, $|\zeta| \leqq 1$, lying in $\mathfrak{R} + i_{kR}{}^3 + \mathfrak{B}_0 + \cdots + \mathfrak{B}_m$ and having its boundary in $\mathfrak{B}_0 + \cdots + \mathfrak{B}_m$. The sum of all the sets of $\{\mathfrak{B}_m\}$ is denoted by $\mathfrak{B}_{kR}$.

The author proves that the function $f(z_1, z_2)$ satisfies the inequality $|f(z_1, z_2)| < M(R)$ in the closure of the full set $\mathfrak{B}_{kR}$. *M. S. Robertson* (New Brunswick, N.J.)

12244:

Baily, Walter L., Jr. On the Hilbert-Siegel modular space. Amer. J. Math. **81** (1959), 846–874.

The symplectic modular group $\Gamma = \mathrm{Sp}(n, Z)$ acts in a well-known manner as a properly discontinuous group of complex-analytic automorphisms on the space S_n of complex symmetric $n \times n$ matrices $Z = X + iY$ with Y positive definite [C. L. Siegel, Math. Ann. **116** (1939),

617–657; MR **1**, 203]; the proof of the theorem that the quotient space S_n/Γ can be "compactified" in such a manner that the result is (complex-analytically isomorphic to) a normal complex projective variety was given in detail by H. Cartan and I. Satake [*Séminaire H. Cartan, 1957/58*, Paris, 1958; MR **21** #2750; especially Lectures 11–17]. More generally, for a totally real algebraic number field K, with $\sigma_1, \cdots, \sigma_p$ being the distinct isomorphisms of K into the real numbers, the group $\mathrm{Sp}(n, K)$ can be imbedded in $(\mathrm{Sp}(n, R))^p$ by the isomorphism i: $M \to (M^{\sigma_1}, \cdots, M^{\sigma_p})$; the imbedded group $\Gamma^* = i(\mathrm{Sp}(n, K))$, called the Hilbert-Siegel modular group, acts as a properly discontinuous group of complex analytic automorphisms on the space $S_n{}^p = S_n \times \cdots \times S_n$ [I. I. Pyateckiĭ-Šapiro, Amer. Math. Soc. Transl. (2) **10** (1958), 13–58; MR **20**, 991b]. In this paper the author proves that the quotient space $S_n{}^p/\Gamma^*$ can also be compactified to yield a normal complex projective variety. The proof closely parallels that of Cartan and Satake [loc. cit.], but the generalization is by no means trivial.

R. C. Gunning (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 12447, B12736, B13129.

12245:

Tschauner, Johann. Zur Erzeugung der Gegenbauerschen Polynome. Math. Z. **75** (1960/61), 1–2.

The author derives a generating function for the Gegenbauer polynomials from their recursion formulae in the usual way. *E. Pinney* (Berkeley, Calif.)

12246:

Lakshmana Rao, S. K. On the relative extrema of the Turán expression for the general Hermite function. Golden Jubilee Research Volume, pp. 200–204. Indian Institute of Science, Bangalore, 1959.

Cet article est la suite d'un article du même auteur, paru en 1958 [Math. Student **26** (1958), 1–6; MR **21** #1409]. Les fonctions d'Hermite considérées, $H_n(x)$, correspondent à n quelconque, mais réel et supérieur ou égal à un. $\Delta_n(x)$ désignant l'expression bien connue de Turán, relative à $H_n(x)$, l'auteur montre que les maximums et les minimums relatifs à $e^{-x^2}\Delta_n(x)$ forment deux suites (finies) croissantes, lorsque x décroit de $+\infty$ à 0. En outre, $M_{r,n}$ et $m_{r,n}$ désignant respectivement le r^e maximum et le r^e minimum de $e^{-x^2}\Delta_n(x)$, comptés à partir de $x = +\infty$, sur $(+\infty, 0)$, $M_{r,n}$ et $m_{r,n}$ sont des fonctions croissantes de n.

Une petite omission, on aimerait savoir pourquoi $H_n(x)$ a $[n]$ racines réelles distinctes et comment se séparent les racines de H_n et de H_{n+1}, ou tout au moins qu'une référence bibliographique précise fût donnée. En outre, les énoncés des théorèmes 4 et 5 doivent être modifiés légèrement. Les démonstrations supposent $x \geqq 0$, il faut donc indiquer que l'on ne considère que l'intervalle $(+\infty, 0)$ et supprimer $[n] \geqq r$. En effet, il existe aussi des extremums relatifs de $e^{-x^2}\Delta_n(x)$ sur l'intervalle $(0, -\infty)$, comme on le voit immédiatement lorsque n est entier positif. *L. Robin* (Paris)

12247:

Weston, V. H. On toroidal wave functions. J. Math. and Phys. **39** (1960/61), 64–71.

Toroidal wave functions are encountered in physical problems involving radiation from a torus or ring, and as a particular case, radiation from a thin circular wire. The two best known functions are $e^{im\phi}V^m_{m+2l}(s, \eta)$ and $e^{im\phi}W^m_{m+2l+1}(s, \eta)$. These are the solutions of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$.

In this paper some simplified forms of these functions are given, together with their auxiliary functions. Asymptotic approximations for them in the far field are derived in detail and prove to be much simpler and more easily handled than the previously known formulae. Also, their asymptotic values in the case of thin rings are held down to smaller wavelengths.

L. J. Slater (Cambridge, England)

12248:

Sim, A. C. An asymptotic expansion for Bessel functions derived with respect to their order. Proc. Cambridge Philos. Soc. **57** (1961), 284–287.

The author constructs an inhomogeneous differential equation satisfied by $\partial^n K_\nu(z)/\partial\nu^n$, and obtains an integral representation satisfied by this function. The integrals again involve Bessel functions: replacement of these by their asymptotic values suggests the form of an asymptotic expansion of $\partial K_\nu(z)/\partial\nu$ (presumably for large z), and the coefficients of this expansion are determined from the differential equation. The expansion terminates if 2ν is an odd integer. *A. Erdélyi* (Pasadena, Calif.)

12249:

Maass, Hans. Zur Theorie der Kugelfunktionen einer Matrixvariablen. Math. Ann. **135** (1958), 391–416.

Let $X = (x_{ij})$ be a real $(m \times n)$-matrix $(m \geq 2n)$. A linear differential operator Ω is a polynomial in the elements of $\partial/\partial X = (\partial/\partial x_{ij})$ whose coefficients are functions of X with derivatives of arbitrary high order. It is said to be "invariant" if it is invariant under the group of mappings $X \to UXV$ where U is an arbitrary orthogonal $(m \times m)$-matrix and V is an arbitrary non-singular $(n \times n)$-matrix. Two operators are said to be equal if $\Omega_1 f = \Omega_2 f$ for all functions $f(X)$ satisfying $f(X) = f(XV)$ for matrices V with determinant $|V| = 1$. All linear differential operators constitute a ring $\mathfrak{R}$ which is generated by the operators $\sigma(X'\partial/\partial X)$, $\sigma(\Lambda^{2\nu})$ $(\nu = 1, 2, \cdots, n)$, where σ denotes the trace and $\Lambda = X\partial/\partial X' - (X\partial/\partial X')'$ [H. Maass, J. Indian Math. Soc. (N.S.) **20** (1956), 117–162; MR **19**, 252]. A function $u(X)$ is said to be a spherical function of type (m, n) if: (1) $u(X)$ is a polynomial with the property $u(XV) = u(X)$ for all V with $|V| = 1$; (2) $u(X)$ is an eigenfunction of the ring $\mathfrak{R}$; (3) $u(X) \not\equiv 0 \pmod{|X'X|}$. Any spherical function is harmonic, i.e., it satisfies $|\partial/\partial X', \partial/\partial X|u(X) = 0$. Since the degree of $u(X)$ is divisible by n it can be represented by nk, where k is an integer. The author studies for a given k the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ defined by

$$\sigma(\Lambda^{2\nu})u(X) = \lambda_\nu u(X) \qquad (\nu = 1, 2, \cdots, n).$$

He determines their possible values explicitly for $n = 2$ by means of the generalized Legendre differential equation. He also proves (for $n = 2$) the uniqueness of the representation (following from the paper cited above):

$$u(X) = \sum_{0 \leq \nu \leq \frac{1}{2}k} |X'X|^\nu u_\nu(X)$$

of any form satisfying $u(XV)=|V|^k u(X)$ in terms of harmonic forms $u_\nu(X)$. Again for $n=2$ the functions $u_k(X)=|L'X|^k$, where L is an arbitrary complex $(m\times n)$-matrix such that $L'L=0$, rank $L=n$, generate the linear set of all spherical functions of degree $2k$ with maximal eigenvalue λ_1. *H. D. Kloosterman* (Leiden)

12250:

Maass, Hans. Zur Theorie der harmonischen Formen. Math. Ann. **137** (1959), 142–149.

The author considers forms (homogeneous polynomials) $u(X)$ in a real variable $(m\times n)$-matrix X $(m>n)$ and satisfying the homogeneity condition $u(XV)=|V|^k u(X)$ if $|V|\neq 0$. The set of these forms is denoted by $\mathfrak{F}_k^{(n)}$ and $\mathfrak{H}_k^{(n)}$ is the subset of harmonic forms satisfying $|\partial/\partial X', \partial/\partial X|u(X)=0$. He proves: (1) Every form $u(X)\in\mathfrak{F}_k^{(n)}$ can be uniquely represented in the form $u(X)=\sum_\nu |X'X|^\nu u_\nu(X)$, where $u_\nu(X)\in\mathfrak{H}_{k-2\nu}^{(n)}$ and $0\leq\nu\leq\frac{1}{2}k$. (2) The set $\mathfrak{H}_k^{(n)}$ consists of the finite sums $\sum|L'X|^k$, where L is an $(m\times n)$-matrix with $|L'L|=0$. In the special case $n=2$ these results were already proved by the author [#12249]. The proofs of the theorems follow from Theorem 3: Let $\mathfrak{x}$ be a column of variables $x_1, x_2, \cdots, x_m$ and let $F_i(\mathfrak{x})$ $(i=1, 2, \cdots, q)$ be a set of non-constant forms with real coefficients. Let $\mathfrak{P}$ be the ideal generated by the $F_i(\mathfrak{x})$, $\mathfrak{N}$ the set of its zeros and suppose that $\mathfrak{P}$ is the set of all polynomials vanishing on $\mathfrak{N}$. Then any form $u(\mathfrak{x})$ can be written as a finite sum: $u(\mathfrak{x})=\sum(F_1(\mathfrak{x}))^{\mu_1}(F_2(\mathfrak{x}))^{\mu_2}\cdots(F_q(\mathfrak{x}))^{\mu_q}(a'\mathfrak{x})^\nu$ with $a\in\mathfrak{N}$. A form $u(\mathfrak{x})$ (satisfying the homogeneity condition $u(\mathfrak{x}a)=a^k u(\mathfrak{x})$ is termed F-harmonic (where F is a non-constant irreducible form of degree $g>0$ say) if $F(\partial/\partial\mathfrak{x})u(\mathfrak{x})=0$. Denote by $\mathfrak{H}_k(F)$ the set of all F-harmonic forms of degree k. Then every form $u(\mathfrak{x})$ of degree k can uniquely be written as a sum $\sum_\nu(F(\mathfrak{x}))^\nu u_\nu(\mathfrak{x})$ with $u_\nu(\mathfrak{x})\in\mathfrak{H}_{k-g\nu}(F)$, $0\leq\nu\leq k/g$. The set $\mathfrak{H}_k(F)$ consists of all finite sums $\sum(a'\mathfrak{x})^k$ with $F(a)=0$. *H. D. Kloosterman* (Leiden)

12251:

Paulauskas, V. On the convergence of series obtained in the solution of integral equations of Lamé functions. Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslų Ser. **4** (1955), 61–66. (Lithuanian. Russian summary)

Author's summary: "A method is presented for determining the radius of convergence of certain series which were previously obtained by the author in determining Lamé functions from integral equations [same Darbai **2** (1954), 5–30]. A solution is sought, by the method of undetermined coefficients, of the differential equation for Lamé functions in the form of a polynomial in Jacobi elliptic functions. The coefficients B_j of the solution depend on the modulus k of the Jacobi functions. The problem reduces to the solution of a system of algebraic equations, from whose secular equation is obtained a parameter A, and also the B_j as power series in k^2. The radius of convergence of these series for an m-term Lamé function is determined as the least modulus of the roots of an algebraic equation of degree $m(m-1)$. It is shown that for four types of Lamé functions the last equation turns out to be recurrent, while to the remaining four types correspond two pairs of equations with the same coefficients but in inverse order."

12252:

Wilcox, Calvin H. Addition theorems for arbitrary plane and spherical waves. Proc. Amer. Math. Soc. **11** (1960), 817–825.

The addition theorems for the solutions of the wave equation $u-\partial^2 u/\partial t^2=0$, the one for plane waves of the form $u=f(t-z)$ and the other for spherical waves of the form $u=f(t-R)/R$ are obtained. The important calculations are based on Hobson's relation for homogeneous polynomials of degree n:

$$H_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)F(r)=H_n(x,y,z)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n F(r),$$

specially applied in the case

$$H_n(x^2+y^2,z)=(-1)^n(n+\tfrac{1}{2})r^n P_n(z/r),$$

where the $P_n(x)$ are the Legendre polynomials.

In the former case, the plane wave $f(t-z)$ is decomposed into partial waves in the following way:

$$f(t-z)=\sum_{n=0}^{\infty}H_n(x^2+y^2,z)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \times[r^{-1}\{I^{n+1}f(t+r)-I^{n+1}f(t-r)\}].$$

The spherical wave $f(t-R)/R$ is decomposed into

$$\frac{f(t-r)}{R}=\sum_{n=0}^{\infty}H_n(x^2+y^2,z)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n a^n\left(\frac{1}{a}\frac{\partial}{\partial a}\right)^n \times[(ra)^{-1}\{I^{2n+1}f(t-r+a)-I^{2n+1}f(t-r-a)\}],\quad r>a.$$

The author succeeded in obtaining the useful formula

$$I^m f(t)=\{(m-1)!\}^{-1}\int_0^t f(t-x)^{m+1}f(x)\,dx$$

by means of which, and together with Hobson's relation, the whole calculations were possible. *T. Inui* (Tokyo)

12253:

MacRobert, T. M. Recurrence formulae for the E-functions. Math. Z. **73** (1960), 254–255.

Three recurrence formulae for the E-functions are obtained, viz.,

$$E\left(\begin{matrix}k+1,\alpha_1,\cdots,\alpha_p\\ k,\rho_1,\cdots,\rho_q\end{matrix}:z\right)=kE(p;\alpha_r:q;\rho_s:z) -z^{-1}E(p;\alpha_{r+1}:q;\rho_{s+1}:z);$$

$$2lE\left(\begin{matrix}k,\tfrac{1}{2}k+1,\quad l\quad,\alpha_1,\cdots,\alpha_p\\ \tfrac{1}{2}k,k-l+1,\rho_1,\cdots,\rho_q\end{matrix}:z\right) =E\left(\begin{matrix}k+1,l+1,\alpha_1,\cdots,\alpha_p\\ k-l+1,\rho_1,\cdots,\rho_q\end{matrix}:z\right) +z^{-1}E\left(\begin{matrix}k+1,l+1,\alpha_1+1,\cdots,\alpha_p+1\\ k-l+1,\rho_1+1,\cdots,\rho_q+1\end{matrix}:z\right);$$

$$kE\left(\begin{matrix}k,\alpha_1,\cdots,\alpha_p\\ k+1,\rho_1,\cdots,\rho_q\end{matrix}:z\right)=E(p;\alpha_r:q;\rho_s:z) +z^{-1}E\left(\begin{matrix}k+1,\alpha_1+1,\cdots,\alpha_p+1\\ k+2,\rho_1+1,\cdots,\rho_q+1\end{matrix}:z\right).$$

They hold for all values of p and q. They are obtained

from recurrence formulae for generalised hypergeometric functions, e.g.,

$$F\left(\begin{matrix}k+1, \alpha_1, \cdots, \alpha_p \\ k, \rho_1, \cdots, \rho_q\end{matrix}; z\right) = F(p; \alpha_r : q; \rho_s : z) + \frac{\alpha_1\alpha_2\cdots\alpha_p}{k\rho_1\rho_2\cdots\rho_q} zF(p; \alpha_r+1 : q; \rho_s+1 : z)$$

($p \leqq q+1$; if $p=q+1$, $|z|<1$) which can be verified by comparing the coefficients of z^n on each side. On replacing z by $-1/z$ and multiplying by $k\{\prod \Gamma(\alpha_r)\}/\{\prod \Gamma(\rho_s)\}$ the first recurrence formula is deduced.

By means of Dougall's second theorem and Dixon's theorem the author obtains from the last relations the special case

$$F\left(\begin{matrix}\alpha+1, \beta+1, \gamma+1 \\ \alpha-\beta+2, \alpha-\gamma+2\end{matrix}; 1\right) = \frac{\Gamma(\alpha-\beta+2)\Gamma(\alpha-\gamma+2)}{2\beta\gamma\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)} \times \left\{\frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\alpha+\frac{1}{2}-\beta-\gamma)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}-\beta)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}-\gamma)} - \frac{\Gamma(\frac{1}{2}\alpha+1)\Gamma(\frac{1}{2}\alpha+1-\beta-\gamma)}{\Gamma(\frac{1}{2}\alpha+1-\beta)\Gamma(\frac{1}{2}\alpha+1-\gamma)}\right\}$$

($\mathrm{Re}\,(\alpha-2\beta-2\gamma) > -1$) and some analogous results.

S. C. van Veen (Delft)

12254:

Meulenbeld, B. The generalized Legendre functions with special relations between the parameters. Math. Z. **75** (1960/61), 192–200.

Generalized Legendre functions are a special notation for certain multiples of hypergeometric series. In the present paper the author expresses in this notation the formulas connecting particular hypergeometric series with the incomplete beta function, elliptic integrals, and Jacobi polynomials. *A. Erdélyi* (Pasadena, Calif.)

12255:

Belardinelli, G. Risoluzione analitica delle equazioni algebriche generali. Rend. Sem. Mat. Fis. Milano **29** (1959), 13–45. (English summary)

Texte d'une conférence faite à Milan en 1958, où l'auteur rappelle et résume les travaux relatifs à la résolution des équations algébriques, depuis ceux de Betti, Brioschi et Hermite, qui remontent à une centaine d'années, jusqu'aux récents résultats obtenus par plusieurs mathématiciens (dont lui-même) et où sont utilisés systématiquement, pour cette résolution, les fonctions hypergéométriques de plusieurs variables.

R. Campbell (Caen)

12256:

Belardinelli, Giuseppe. ★Fonctions hypergéométriques de plusieurs variables et résolution analytique des équations algébriques générales. Mémor. Sci. Math., Fasc. 145. Gauthier-Villars, Paris, 1960. 74 pp. 15 NF; $3.25.

The principal subject of this monograph is the representation of the roots of an algebraic equation as an analytic function of the coefficients. This is a comparatively little known branch of analysis in the development of which the author played an important role.

Part One (nearly one half) of the monograph contains a brief descriptive account of hypergeometric functions. Contents: Chapter I: Gauss' hypergeometric function. Chapter II: "Generalized" hypergeometric functions. Chapter III: Hypergeometric functions of several variables (Appell's and Lauricella's functions, Horn's series, Kampé de Fériet's series).

Part Two is devoted to the "analytic" solution of algebraic equations. Chapter IV: Trinomial equations. Historical notes, integral representations of roots, roots of the trinomial equations $y^n+y^{n-s}+x=0$ and $y^n+uy^{n-s}+1=0$ as functions of x and u, Mellin's integral representation. Chapter V: The general algebraic equation. Roots of $y^n+x_1y^{n_1}+\cdots+x_py^{n_p}-1=0$ as functions of $x_1, \cdots, x_p$ can be represented as multiple Mellin-Barnes integrals and are thus hypergeometric functions. Expansions of Lagrange and Capelli, Birkeland's theorem.

A ten-page bibliography complements the account.

A. Erdélyi (Pasadena, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 12355, 12538, B12712, B12720, B12776.

12257:

Лефшец, С. [Lefschetz, Solomon]. ★Геометрическая теория дифференциальных уравнений [Differential equations: geometric theory]. Translated from the English by R. E. Vinograd and M. I. Grabar'; edited by V. V. Nemyckiĭ and V. M. Alekseev. Izdat. Inostr. Lit., Moscow, 1961. 387 pp. 1.59 r.

A translation of the author's 1957 treatise [Interscience, New York, 1957; MR **20** #1005]. In many instances, the editors have added footnotes in which they amplify what they consider to be obscure points in the original English version.

12258:

Nemytskii, V. V.; Stepanov, V. V. ★Qualitative theory of differential equations. Princeton Mathematical Series, No. 22. Princeton University Press, Princeton, N.J., 1960. viii+523 pp. $12.50.

The book is a translation of material that has already been reviewed previously [Nemyckiĭ and Stepanov, *Kačestvennaya teoriya differencial'nyh urovneniĭ*, OGIZ, Moscow, 1947; MR **10**, 612; Nemyckiĭ, Uspehi Mat. Nauk **4** (1949), no. 6 (34), 91–153; Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 8, 19–39; MR **11**, 526; **14**, 753]. *M. M. Peixoto* (Rio de Janeiro)

12259:

Corduneanu, C. Une application de théorème du point fixe à la théorie des équations différentielles. An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 43–47. (Romanian and Russian summaries)

Let C be the linear space of continuous functions on $-\infty < t < \infty$ to R^n, n-dimensional Euclidean space, with topology given by convergence on compact subsets. Let A be a closed convex subset of C. Let T be a continuous operator with TA a compact subset of A. The Schauder fixed-point theorem applies. In the paper $(Tu)(t) = x_0 + \int_{t_0}^{t} f(t, u(t))dt$ and $A = \{u \mid \|u(t)\| < M(t)\}$, where $M(t)$ is defined as the integral of a bounded summable dominant for $|f(t, x)|$. *D. G. Bourgin* (Urbana, Ill.)

12260:

Langenhop, C. E. Bounds on the norm of a solution of a general differential equation. Proc. Amer. Math. Soc. **11** (1960), 795–799.

L'auteur considère le système (i) $dz/dx = F(x, z)$ où: (1) x est une variable réelle, z et F sont des vecteurs à n composantes complexes; (2) F est continu par rapport à (x, z) pour $x \in [a, b]$ et $\|z\| < +\infty$; (3) $\|F(x, z)\| \leq v(x)g(\|z\|)$; (4) $v(x) \geq 0$ est continue pour $x \in [a, b]$; (5) $g(u)$ est continue et non décroissante sur $[0, \infty)$, $g(u) > 0$ pour $u > 0$.

Toute solution $z(x)$ du système (i), définie sur $[a, b]$ et telle que $\|z(x)\| > 0$, satisfait aux limitations

$$\text{(ii)} \qquad \|z(x)\| \leq G^{-1}(G(\|z(a)\|) + \int_a^x v(s)\,ds),$$

$$\text{(iii)} \qquad \|z(x)\| \geq G^{-1}(G(\|z(a)\|) - \int_a^x v(s)\,ds),$$

où $G(u) = \int_{u_0}^{u} [g(t)]^{-1}dt$, $u_0 > 0$. (ii) est valable sur $[a, b]$ tandis que (iii) est valable pour tous les $x \in [a, b]$, pour lesquels $G(\|z(a)\|) - \int_a^x v(s)ds$ est dans le domaine de définition de G^{-1}.

La limitation (ii) est une conséquence d'un lemme du à I. Bihari [Acta Math. Acad. Sci. Hungar. **7** (1956), 81–94; MR **18**, 38] et de l'inégalité

$$\|z(x)\| \leq \|z(a)\| + \int_a^x v(s)g(\|z(s)\|)\,ds.$$

On démontre que (iii) n'est pas la conséquence de l'inégalité

$$\|z(x)\| \geq \|z(a)\| - \int_a^x v(s)g(\|z(s)\|)\,ds.$$

C. Corduneanu (Iaşi)

12261:

Kuz'min, P. A. Quadratic integrals of linear mechanical systems. Prikl. Mat. Meh. **24** (1960), 575–577 (Russian); translated as J. Appl. Math. Mech. **24**, 856–860.

Es wird untersucht unter welchen Bedingungen ein System linearer Differentialgleichungen mit konstanten Koeffizienten quadratische Integrale von der Form $V = \sum_1^{2k} a_{ij}x_ix_j$ besitzt. Derartige Integrale interessieren vor allem bei der Untersuchung der Stabilität dynamischer Systeme. Durch geschickte Transformation der Variablen läßt sich das System der Ausgangsgleichungen in eine kanonische Form transformieren, bei der die darin vorkommende "Hamilton-Funktion" als Summe aller für das Ausgangssystem existierenden quadratischen Integrale erscheint. Nach den Sätzen von Ljapunov ist es notwendig und hinreichend für Stabilität des Systems, wenn diese Funktion positiv definit ist.

K. Magnus (Stuttgart)

12262a:

Karim, Rahmi Ibrahim Ibrahim Abdel. Über den Resonanzbegriff bei Systemen von n linearen gewöhnlichen Differentialgleichungen erster Ordnung. Arch. Rational Mech. Anal. **7** (1961), 21–28.

12262b:

Karim, Rahmi Ibrahim Ibrahim Abdel. Über den Resonanzfall bei Systemen von n nichtlinearen gewöhnlichen Differentialgleichungen erster Ordnung. Arch. Rational Mech. Anal. **7** (1961), 29–36.

In the first paper the author shows that the equation (1) $x' = A(t)x + f(t)$, with A, f continuous and of period p, has bounded solutions solely if the equation (2) $z' = -A^*(t)z$ has either a solution with period p such that $\int_0^p z^*(s)\cdot f(s)ds = 0$, or no periodic solution at all. {Of eleven theorems, six are elementary and found in textbooks; three hinge on the well-known relation

$$z^*(t)\cdot x(t) = z^*(0)\cdot x(0) + \int_0^t z^*(s)\cdot f(s)\,ds,$$

where x, z are any solutions of (1), (2), respectively; and the remaining two are immediate corollaries of the latter three.}

In the second paper the author applies these results to the equation $x' = g(x, t) + h(t) + \beta f(t)$, with g, h, f continuous and of period p in t and $|\beta|$ small, by considering the variational equation relative to a solution with period p of the equation $u' = g(u, t) + h(t)$.

H. A. Antosiewicz (Los Angeles, Calif.)

12263:

R.-Salinas, Baltasar. Variation of the characteristic roots of a second order differential equation with periodic coefficients. Ann. Mat. Pura Appl. (4) **52** (1960), 107–161. (Spanish. English summary)

From the author's summary: "The range of variation of the characteristic roots of the second order differential equation $y'' + qy = 0$ is determined when q ranges over the family of measurable periodic functions of period ω satisfying the inequalities $a \leq q(x) \leq b$ and $\sigma_1 \leq \int_0^\omega q(x)dx \leq \sigma_2$." {Slight corrections by the reviewer.} Actually, the extrema of the coefficient of the middle term of the characteristic equation are determined precisely (q is supposed to be essentially bounded even if a, b are not both finite); they turn out to be among a finite set of numbers depending in particular on the maximum and minimum number of zeros of a non-trivial solution within one period. Criteria for stability for all q in the described family are derived. There is an extension to the equation $y'' + py' + qy = 0$, the interest of which lies less in the actual result (the description of the family in which q varies becomes clumsy) than in the neat reduction to the previous case by means of a change of both independent and dependent variable, with preservation of periodicity; this is an improvement over a previous paper by the author [Rend. Circ. Mat. Palermo (2) **8** (1959), 206–224; MR **22** #6909].

J. J. Schäffer (Montevideo)

12264:

Hale, Jack K. On the characteristic exponents of linear periodic differential systems. Bol. Soc. Mat. Mexicana (2) **5** (1960), 58–66.

The present paper represents a new exposition concerning linear systems with periodic coefficients

$$y' = Ay + \varepsilon C(t)y \qquad (y = y_1, y_2, \cdots, y_N)$$

near systems with constant coefficients. In principle, of course, the behavior of the solutions is described by Floquet's theory, but this description will depend on the nature of the characteristic multipliers. Here symmetries, reversibility and similar properties are shown to lead to

criteria which ensure that all or some of the characteristic multipliers lie on the unit circle, in which case one obtains boundedness for all or some solutions. The paper relies on a method of successive approximation (described in several previous papers, e.g., Bailey and Cesari, Arch. Rational Mech. Anal. **1** (1958), 246–271 [MR **20** #3322] and the author, Illinois J. Math. **2** (1958), 586–592 [MR **20** #3323] and in L. Cesari's book on *Asymptotic behavior and stability problems in ordinary differential equations*, Springer, Verlag, 1959; MR **22** #9673]. In the last section, a theorem implying asymptotic stability for weakly nonlinear systems of differential equations is mentioned.

J. Moser (New Rochelle, N.Y.)

12265:

Villari, Gaetano. Contributi allo studio asintotico dell'equazione $x''(t)+p(t)x(t)=0$. Ann. Mat. Pura Appl. (4) **51** (1960), 301–328.

Der Verfasser betrachtet das asymptotische Verhalten der Lösungen der Differentialgleichung (1) $x''+p(t)x=0$ sowie der adjungierten Gleichung $y''-p(t)y=0$, wobei der Koeffizient $p(t)$ asymptotisch stetig und nicht negativ ist und auch nicht abschnittsweise identisch verschwindet. Vom Sonderfall $p(t)=$konstant ausgehend teilt der Verfasser die Lösungen in je zwei Klassen ein:

A. Lösungen $x_A(t)$, für die asymptotisch $x_A'(t)x_A''(t)<0$.
B. Lösungen $x_B(t)$, bei denen zu beliebigem $T>0$ ein Wert $t_0>T$ existiert, so daß $x_B'(t_0)x_B''(t_0)\geqq 0$.
C. Lösungen $y_C(t)$, bei denen zu beliebigem $T>0$ ein Wert $t_0>T$ existiert, so daß $y_C'(t_0)y_C''(t_0)\leqq 0$.
D. Lösungen $y_D(t)$, für die asymptotisch $y_D'(t)y_D''(t)>0$.

In einer Anzahl von Sätzen werden folgende Eigenschaften der Lösungen bewiesen: Die Lösungen $x_A(t)$ bilden einen linearen Raum der Dimension 1 und sind asymptotisch beschränkt und monoton, d.h., (3) $\lim_{t\to\infty}|x_A(t)|=c\geqq 0$, $\lim_{t\to\infty}x_A'(t)=\lim_{t\to\infty}x_A''(t)=0$. Die Lösungen $x_B(t)$ sind entweder alle oszillatorisch oder alle nicht oszillatorisch und unbeschränkt. Die Lösungen $y_C(t)$ sind entweder alle oszillatorisch oder alle nicht oszillatorisch und bilden einen linearen Raum der Dimension 2. Die Lösungen $y_D(t)$ sind alle asymptotisch unbeschränkt und monoton. Die Lösungen $y_C(t)$ sind in dem Sinne orthogonal zu den Lösungen $x_A(t)$, daß $x_A(t)y_C''(t)-x_A'(t)y_C'(t)+x_A''(t)y_C(t)=0$. Hinsichtlich der oszillatorischen Lösungen zeigt der Verfasser, daß aus der Existenz oszillatorischer Lösungen von (1) (Klasse B) auch die Existenz solcher Lösungen von (2) (Klasse C) folgt und umgekehrt. Weiterhin studiert der Verfasser das asymptotische Verhalten in dem Fall, wenn keine oszillatorischen Lösungen auftreten, und insbesondere die Stabilität der Nullösung. Allerdings ist nur bedingte Stabilität möglich, d.h., Stabilität innerhalb einer Lösungsmannigfaltigkeit, die einen linearen Raum bildet. Sie ist bei der Gleichung (1) dann und nur dann asymptotisch, d.h., (3) gilt mit $c=0$, wenn (4) $\lim_{t\to\infty}\int_{t_0}^{t}\tau^2 p(\tau)d\tau=+\infty$; bei der Gleichung (2) herrscht unter dieser Bedingung totale Instabilität, so daß alle Lösungen außer der trivialen unbeschränkt sind. Ist der Grenzwert aber endlich, so enthält die Klasse C eine Unterklasse der Dimension 1, für die (3) erfüllt ist; die triviale Lösung von (2) ist dann bedingt stabil, aber nicht asymptotisch. Die Bedingung (4) ist notwendig für die Existenz oszillatorischer Lösungen von (1) und (2); hinreichend ist die Bedingung $\lim_{t\to\infty}\int_{t_0}^{t}\tau^{2-\lambda}p(\tau)d\tau=+\infty$, wobei $\lambda>0$.

R. Reissig (Berlin)

12266:

Azbelev, N. V.; Calyuk, Z. B. On the question of the distribution of the zeros of solutions of a third-order linear differential equation. Mat. Sb. (N.S.) **51** (**93**) (1960), 475–486. (Russian)

The authors consider the equation

$$(*)\qquad L[y]=y'''+p_2(x)y''+p_1(x)y'+p_0(x)y=0,$$

where the coefficients and their derivatives (necessary for the existence of the adjoint operator $-L^*[y]$) are continuous on $[a,\infty)$. Two consecutive zeros of a solution of the equation (*) form an (i,k)-pair if the multiplicities are not less than i and k respectively. The interval $[t,b)\subset[a,\infty)$ is called an (i,k)-interval of non-oscillation if no solution of (*) has (i,k)-pairs of zeros in this interval. Evidently, only the cases $i,k=1,2$ are interesting. Necessary and sufficient conditions are given for an interval to be an (i,k)-interval of non-oscillation. A typical result of the paper is the following: the interval $[t,b)$ is a $(1,1)$-interval of non-oscillation if and only if there exist two functions $v(x)$ and $v^*(x)$ of class C^3 on $[t,b)$, such that $v(t)=v^*(t)=0$, $v(x)>0$, $v^*(x)>0$, $L[v]\leqq 0$, $L^*[v]\leqq 0$ for $x\in(t,b)$.

C. Corduneanu (Iaşi)

12267:

Zimmerberg, Hyman J. Two-point boundary problems involving a parameter linearly. Illinois J. Math. **4** (1960), 593–608.

The paper deals with extensions of various concepts of Bliss [Trans. Amer. Math. Soc. **28** (1926), 561–584; **44** (1938), 413–428] and Reid [ibid. **52** (1942), 381–425; Illinois J. Math. **2** (1958), 434–453; MR **4**, 100; **20** #3331] to the boundary problem on $a\leqq x\leqq b$ determined by the differential equation system $y'-A(x)y=\lambda B(x)y$ and boundary condition $(M_0+\lambda M_1)y(a)+(N_0+\lambda N_1)y(b)=0$. Here $A(x)$ and $B(x)$ are complex-valued and continuous $n\times n$ matrices on $a\leqq x\leqq b$, $B(x)\not\equiv 0$, and M_0, N_0, M_1, N_1, are complex-valued $n\times n$ coefficient matrices for which the $n\times 2n$ matrix $\|M_0+\lambda M_1 \quad N_0+\lambda N_1\|$ is of rank n for every complex λ. The notions of adjointness, equivalence, normality and abnormality, symmetrizability, and definiteness are considered.

C. R. Putnam (Lafayette, Ind.)

12268:

Gloistehn, H. H. Fehlerabschätzungen aus den Defekten bei Randwertaufgaben gewöhnlicher und partieller Differentialgleichungen 2. Ordnung. Arch. Rational Mech. Anal. **7** (1961), 135–142.

Consider the boundary value problem

$$(1)\qquad \begin{aligned} Tu &\equiv -\sum_{j,k=1}^{n}a_{jk}(x)u_{jk}+F(x,u,u_i)=r(x) \quad \text{in } B,\\ U_\alpha u &\equiv u-\alpha\frac{\partial u}{\partial\sigma}=f_\alpha \quad \text{on } \Gamma \quad (\alpha\geqq 0), \end{aligned}$$

where Γ denotes the boundary of a bounded domain B in the n-dimensional x-space and a_{jk} is positive definite in B. For a solution $u(x)$ of (1) and any approximation $v(x)$ of u, the author derives estimates for the error-function $(u-v)/\varphi$ where φ satisfies the conditions $\varphi>0$ and

$$-\sum_{j,k}a_{jk}\varphi_{jk}+\frac{d}{dt}[F(x,v+t\varphi,v_i+t\varphi_i)]_{t=0}>0.$$

The estimates are essentially based on the assumption that functions $N_1(x) \leqq 0 \leqq N_2(x)$ can be found such that

$$N_1(x)z^2 \leqq F(x, v+\varphi z, v_t+\varphi_t z) - F(x, v, v_t)$$
$$-\frac{d}{dt}[F(x, v+t\varphi z, v_t+t\varphi_t z)]_{t=0} \leqq N_2(x)z^2.$$

The result is applied to the cases when v satisfies the differential equation or the boundary conditions exactly. In addition, a numerical example is given.

W. C. Rheinboldt (Syracuse, N.Y.)

12269:

Garner, J. B.; Burton, L. P. Solutions of linear differential systems satisfying boundary conditions in the large. Proc. Amer. Math. Soc. **12** (1961), 100–106.

Let the $n \times n$ matrix $A(x)$ be continuous on $[\alpha_1, \alpha_2]$. Let $y(x) = \text{col}\,(y_1(x), \cdots, y_n(x))$ and $b(x) = \text{col}\,(b_1(x), \cdots, b_n(x))$. The authors prove that, if the signs of the $a_{ij}(x)$ $(i \neq j)$ are properly restricted, the non-homogeneous boundary value problem $y' = Ay + b$, $y_i(\alpha_1) = \beta_i$ $(i < n)$, $y_n(\alpha_2) = \beta_n$ has a unique solution; there is a dual theorem if $\alpha_2 < \alpha_1$. If $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$, under rather complicated conditions on the signs of the $a_{ij}(x)$, it is proved that the problem $y' = Ay + b$, $y_1(\alpha_1) = \beta_1$, $y_i(\alpha_2) = \beta_i$ $(1 < i < n)$, $y_n(\alpha_3) = \beta_n$, has a unique solution. In the sense that they apply over the entire interval on which the various hypotheses on $A(x)$ are imposed, the results are non-local.

W. J. Coles (Salt Lake City, Utah)

12270:

Rubinowicz, A. "Umgeordnete" und zweiparametrige Eigenwertprobleme, die mit Hilfe der Polynommethode lösbar sind. Acta Phys. Polon. **19** (1960), 533–558.

In this paper the author treats a certain eigenvalue problem of the equation

$$(1) \qquad \frac{d}{dx}\left(P\frac{df}{dx}\right) - \{q(x) - P(x)\}f = 0,$$

using the polynomial method.

If the given equation contains another parameter m besides the eigenvalue parameter λ, the transformed eigenvalue problem of both linear and quadratic type in connection with the parameter m is established. The solution of the latter problem is given if one takes advantage of the solution of the corresponding original eigenvalue problem. By this technique, a number of orthogonal relations in the same domain, but with different density functions, are obtainable. If the eigenvalue problem of the linear transformed equation is solved, its result is useful in treating the linear two-parameter eigenvalue problem of type

$$(2) \qquad \frac{d}{dx}\left(P\frac{df}{dx}\right) - (q(x) - \lambda^{(1)}\rho_1(x) - \lambda^{(2)}\rho_2(x))f = 0.$$

Special results are given in the rather important examples, including Legendre associated functions $P_l^m(x)$ and the generalized functions $P_l^{m,n}(x)$ and radial atomic orbital functions of wave mechanics $R_{nl}(x)$. Difficulties in quadratic eigenvalue problems are stated. In connection with the factorization method of Schrödinger and Infeld, the author proposes to consider the factorization parameter m as an eigenvalue parameter of the transformed problem.

Further, it is shown that in the eigenvalue problems which are solvable by the factorization method, the eigenvalues always form subsequences of the same sequences of numbers, independent of the value of m, which can change by an integer. *T. Inui* (Tokyo)

12271:

Harris, W. A., Jr. Singular perturbations of eigenvalue problems. Arch. Rational Mech. Anal. **7** (1961), 224–241.

This paper is concerned with the singular perturbation theory for an eigenvalue problem for a system of ordinary differential equations on a finite interval $[a, b]$, written in the form

$$(1) \qquad L[y, \alpha] = dy/dt - [\alpha A(t) + B(t, \alpha)]y = \lambda R(t)y,$$

$$(2) \qquad s[y, \alpha] = M(\alpha)y(a, \alpha) + N(\alpha)y(b, \alpha) = 0.$$

Here A, B, R, M and N are square matrices of order $n+m$, y is a vector of order $n+m$, α is a large parameter, and B, M, N are assumed to be convergent power series in α^{-1} for large α. A has the form $\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}$, where A_{22} is of order m. Thus the system (1) degenerates for $\alpha \to +\infty$ into a differential system of n equations for n unknowns. The problem is to find an appropriate system of boundary conditions for the degenerate problem, compatible with $s[y, \infty]$, such that the eigenvalues $\lambda(\alpha)$ and the eigenfunctions $y(t, \alpha)$ of (1), (2) tend for $\alpha \to +\infty$ to the corresponding ones for the degenerate system. Such a system is given under certain assumptions on the matrices A, M, N (which are rather complicated and include, in particular, that A_{22} be diagonalizable with eigenvalues with nonzero real parts). Further, the possibility of asymptotic expansions for $\lambda(\alpha)$ and $y(t, \alpha)$ in powers of α^{-1} is proved for the simple eigenvalues of the degenerate problem. The result is applied to single equations of the form $(P + \varepsilon^{2n-2m}Q)x = \lambda x$, $n > m$, where P, Q are linear differential operators of order $2m$, $2n$ respectively. *T. Kato* (Tokyo)

12272:

Tatarkiewicz, Krzysztof. Sur la résonance de second espèce. Ann. Univ. Mariae Curie-Skłodowska Sect. A **13** (1959), 33–74. (Polish and Russian summaries)

A second-order differential equation (1) $x'' = f(t, x, x')$ is considered. It is assumed that the function $f(t, x, z)$ is continuous and (1) has the uniqueness property. Besides, the following assumption appeared. There exist constants $a_2 \leqq a_1$ and $b_2 \leqq b_1$ such that

$$(2) \qquad 2a_i z + b_j x \leqq f(t, x, z) \leqq 2a_k z + b_l x,$$

where $i = j = 2$; $k = l = 1$ if $x \geqq 0$, $z \geqq 0$; $i = l = 1$, $j = k = 2$ if $x \geqq 0$, $z \leqq 0$; $i = l = 2$, $j = k = 1$ if $x \leqq 0$, $z \geqq 0$ and $i = k = 1$, $k = l = 2$ if $x \leqq 0$, $z \leqq 0$.

The main result of this paper concerns the case that $a_2^2 + b_1 < 0$, $a_1 < 0$, and consists of finding an equation, too complicated to be stated here, which defines explicitly a function $B(a_1, b_1)$ such that if (3) $b_2 > B(a_1, b_1)$, then all nontrivial solutions of (1) and their derivatives oscillate and tend exponentially to zero as $t \to +\infty$ and, if (3) is not satisfied, the author shows there may exist nontrivial solutions of (1) which does not tend to zero as $t \to +\infty$. Also the case $a_2 > 0$ and $a_1^2 + b_1 < 0$ is discussed and

conditions are found which guarantee that each nontrivial solution of (1) is unbounded. *C. Olech* (Baltimore, Md.)

12273:

Tatarkiewicz, Krzysztof. Sur une équation généralisant les équations linéaires avec second membre. Ann. Univ. Mariae Curie-Skłodowska Sect. A **13** (1959), 75–85. (Polish and Russian summaries)

In this paper equation (1) of the previous review is considered, but instead of (2), the author works with the assumption

$$2a_1z+b_1x-N \leqq f(t,x,z) \leqq 2a_2z+b_2x+N,$$

where a_1, a_2, b_1, b_2, N are constants and the configuration of indices in the above inequality is similar to that of the previous paper.

It is shown that (3) [cf. #12272] guarantees the boundedness of solutions of (1) and their derivatives. Also, the case where there are unbounded solutions of (1) is discussed. *C. Olech* (Baltimore, Md.)

12274:

Tatarkiewicz, Krzysztof. Sur les mouvements sous l'influence des forces élastiques généralisées. Ann. Univ. Mariae Curie-Skłodowska Sect. A **13** (1959), 87–114. (Polish and Russian summaries)

In this paper the author extends his results of a previous paper [see review #12272] to the case that $a_2^2+b_1$ is not necessarily negative but $a_1^2+b_1<0$ and $a_1<0$. The results in this case are similar (the same function $B(a_1, b_1)$ appeared) in that the solutions tend to zero but not necessarily exponentially and do not have to oscillate. Also, the non-oscillatory case is discussed; that is, the case when $a_1^2+b_2>0$, $a_1<0$. [For related results, see the reviewer's papers in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 573–579; **7** (1959), 319–326, 487–494; MR **20** #6566; **21** #7326; **22** #4850.]

In the second section of the paper the author gives some examples, and in the third he presents a classification of results concerning the asymptotic behavior of solutions of second-order ordinary differential equations.

C. Olech (Baltimore, Md.)

12275:

Graffi, Dario. Alcuni risultati qualitativi e quantitativi sulle equazioni differenziali non lineari. Bol. Soc. Mat. Mexicana (2) **5** (1960), 13–23.

L'autore espone dapprima alcune sue interessanti ricerche atte a stabilire, in opportune ipotesi, il carattere oscillatorio degli integrali della così detta equazione di Liénard-Mathieu $\ddot{x}+f(x)\dot{x}+[\omega^2+g(x)\sin(\Omega t+\gamma)]x=0$ e alcune limitazioni inferiori per la distanza di due zeri consecutivi. Seguono alcune considerazioni dello stesso autore sull'applicazione del metodo di Kryloff-Bogoliuboff all'equazione $\ddot{x}+\varepsilon f(x,\dot{x})+x=0$ dove ε è un parametro infinitesimo.

L'autore richiama infine alcune ricerche di L. Minozzi [Boll. Un. Mat. Ital. (3) **9** (1954), 196–198; MR **16**, 250], G. Colombo [Rend. Sem. Mat. Univ. Padova **22** (1953), 123–133, 380–398; **23** (1954), 407–421; MR **15**, 128, 427; **16**, 477] sulle soluzioni sottoarmoniche di equazioni del tipo $\ddot{x}+P(x)=A\cos nt$ dove $P(x)$ è un polinomio in x, di G. Aymerich [Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 51–62, 171–186; MR **17**, 39, 369] sull'oscillatore di Rocard e di G. Sansone sull'equazione del pendolo generalizzata $\ddot{\theta}+f(\theta,\alpha)\dot{\theta}=g(\theta)$. *G. Sansone* (Florence)

12276:

Ezeilo, J. O. C. On the existence of periodic solutions of a certain third-order differential equation. Proc. Cambridge Philos. Soc. **56** (1960), 381–389.

The paper deals with the equation (1) $\dddot{x}+a\ddot{x}+b\dot{x}+h(x)=p(t)$, where a, b are constants, $p(t)$ is continuous and periodic with least period ω and the dots indicate differentiation with respect to t. Assuming that a certain set of relations is satisfied by a, b, h, p, it is proved by means of Brouwer's fixed-point theorem that there exists an integer $N_0\geqq 1$ such that for every $n\geqq N_0$ there exists a solution of (1) with period $n\omega$. Besides, if there is no solution having a period ω then for every prime $q\geqq N_0$ there is a solution with a least period $q\omega$. Using the Leray-Schauder technique, under certain conditions on a, b, h, p, one can guarantee the existence of one solution of (1) with least period ω. *M. M. Peixoto* (Rio de Janeiro)

12277:

Opial, Z. Sur un théorème de C. E. Langenhop et G. Seifert. Ann. Polon. Math. **9** (1960/61), 145–155.

In the equation (1) $x''+f(x)x'+g(x)=p(t)$ $(x'=dx/dt)$, let f, g, and p be real-valued functions defined and continuous for all real values of their arguments. Suppose for x on the closed interval $[a, b]$, where $a<b$, that $f(x)\geqq m>0$, where g is strictly increasing, g' is continuous and (2) $0\leqq g'(x)\leqq l<\lambda_0 m^2$, where λ_0 is the unique root of the equation (3) $\ln\lambda=[\pi+2\operatorname{arc\,cot}(4\lambda-1)^{1/2}]/(4\lambda-1)^{1/2}$. Under these assumptions, the author proves that (1) cannot have more than one solution x satisfying $a\leqq x(t)\leqq b$ for all real t. Using this result, he is able to weaken the restriction $l<m^2$ imposed by Langenhop and Seifert [Proc. Amer. Math. Soc. **10** (1959), 425–432; MR **21** #3623] in proving the existence, uniqueness and asymptotic stability of an almost periodic solution of (1) when p is almost periodic since λ_0 is approximately 3. He also points out that similar results are readily established if instead of (2) one has $\lambda_1 m^2<g'(x)<\lambda_2 m^2$ for $a\leqq x\leqq b$, where $\lambda_2>0$ and $\lambda_1>0$ are related through an appropriate transcendental equation in place of (3).

C. E. Langenhop (Princeton, N.J.)

12278:

Markus, Lawrence. The behavior of the solutions of a differential system near a periodic solution. Ann. of Math. (2) **72** (1960), 245–266.

Let M^n be a differentiable manifold and $\mathscr{S}$ a differential equation defined on it. The paper contains a detailed investigation of the behavior of the trajectories of $\mathscr{S}$ near a closed orbit S. The orbit S is said to be elementary in case its transversal germ has a differential (i.e., linear part) having no eigenvalue of modulus one; S is said to be isolated if it has a tubular neighborhood N such that within N, S is the only closed orbit of S. Theorem: If S is elementary, then it is isolated. A good deal of the paper is concerned with closed orbits exhibiting linear transversal germs. It is shown that given any $(n-1)\times(n-1)$ matrix C one can always find a differential system defined in either a solid anchor-ring or Klein bottle and having a

closed orbit whose transversal germ is a linear transformation corresponding to C. Another result is that given a system $\mathscr{S}$ with a closed orbit S, one can, by an arbitrarily small C^1-small modification of $\mathscr{S}$, obtain another system $\mathscr{S}'$ for which S is an elementary closed orbit with distinct, algebraically independent (over the rationals) eigenvalues. The paper ends with some interesting results for the case that M^n and $\mathscr{S}$ are real-analytic. Theorem: Given a closed orbit S of $\mathscr{S}$ and a positive integer k, then there exists a tubular neighborhood N_k of S such that the set of all closed orbits of $\mathscr{S}$ contained in N_k, which make k turns around S, lies on a 2-dimensional analytic submanifold of M^n. If $n=3$ and if, for a certain k, there exist closed orbits of $\mathscr{S}$ making k turns around S and arbitrarily close to S, then any other closed orbit of $\mathscr{S}$ sufficiently near S makes k turns around S. *M. M. Peixoto* (Rio de Janeiro)

12279:

Hartman, Philip. A lemma in the theory of structural stability of differential equations. Proc. Amer. Math. Soc. **11** (1960), 610–620.

Consider the system of real differential equations (1) $dx/dt=\Gamma x+F(x)$ where x is an n-vector, $F(x)$ is of class C^2 with $F(x)=o(|x|)$ with $|x|$ being the Euclidean length of x, and Γ is a constant matrix with eigenvalues having non-zero real part. The author proves that there is a topological mapping $R: u=u(x)$ of a neighborhood of $x=0$ onto a neighborhood of $u=0$ such that the solution paths of (1) are mapped onto solution paths of $u'=\Gamma u$. This result implies that n-dimensional saddle points are structurally stable. The mapping R in general is not of class C^1 even when F is analytic.

M. M. Peixoto (Rio de Janeiro)

12280:

Roxin, Emilio O. Reachable zones in autonomous differential systems. Bol. Soc. Mat. Mexicana (2) **5** (1960), 125–135.

Se t è la variabile indipendente e x, $f(x)$, $w(x,t)$ sono vettori $n\times 1$, l'Autore considera l'equazione $\dot{x}=f(x)$ [$\dot{x}=dx/dt$] e l'equazione perturbata (*) $\dot{x}=f(x)+w(x,t)$.

Fissato un punto x_0 l'Autore dice che un punto x_1 è raggiungibile da x_0 rispetto ad una classe di funzioni perturbatrici $w(x,t)$ se considerando la soluzione $x(t)$ della (*) che soddisfa la condizione inziale $x(0)=x_0$ esiste un $t_1>0$ tale che $x(t_1)=x_1$. L'Autore studia l'insieme dei punti raggiungibili nel caso che $w(x,t)$ varii in una classe di funzioni di norma euclidea limitata e nel caso di $n=2$ se $w(x,t)=\varphi(x)\alpha(t)$ con $\varphi(x)$ fissa e $\alpha(t)$ funzione continua arbitraria. *G. Sansone* (Florence)

12281:

Opial, Z. Sur les périodes des solutions de l'équation différentielle $x''+g(x)=0$. Ann. Polon. Math. **10** (1961), 49–72.

Consider the differential equation $x''+g(x)=0$ with $g(x)$ continuous on the real line, $xg(x)>0$ for $x\neq 0$. Set $G(x)=\int_0^x g(s)ds$. For each $x\neq 0$, let $x(t)$ be the (periodic) solution with $x(0)=x$, $x'(0)=0$, and let $(-T_g(x), T_g(x))$ be the largest interval in which $x(t)$ does not vanish; $T_g(x)$ may be called the positive or negative "half-period" of $x(t)$, according to the sign of x; if $g(x)$ is odd, $T_g(x)$ is actually half the period of $x(t)$. If h, H are put in place of g, G, then $T_h(x)$ is defined similarly. The following is a sample of the results in the paper: $T_g(x)\leqq T_h(x)$ for all $x\neq 0$ implies $G(x)\geqq H(x)$ for all x, and is implied by $|g(x)|\geqq|h(x)|$ for all x; equality holds only simultaneously; $T_g(x)$ is independent of x if and only if g is linear; $\lim T_g(x)/T_h(x)=L$ implies $\lim H(x)/G(x)=L^2$ and is implied by $\lim h(x)/g(x)=L^2$, $0\leqq L\leqq\infty$, where $x\to+\infty$ or $-\infty$ in the first implication, and $x\to+\infty$, $-\infty$, or 0 in the other; corresponding inequalities are established for the lim sup and lim inf.

J. J. Schäffer (Montevideo)

12282:

Grobman, D. M. Homeomorphism of systems of differential equations. Dokl. Akad. Nauk SSSR **128** (1959), 880–881. (Russian)

From the introduction: "V. V. Nemyckiĭ stated the following problem: to find conditions under which the topological structure of a neighborhood of a stationary point of an almost linear system of differential equations is determined by the corresponding linear system. For three equations this problem was solved by R. M. Minc [Nauč. Dokl. Vyšš. Školy. Fiz.-Mat. Nauki **1** (1958)]. In the present article the solution is given for n equations."

12283:

Šestakov, A. A. Asymptotic behavior of solutions of multidimensional systems of ordinary differential equations with a singular point of higher order. Dokl. Akad. Nauk SSSR **131** (1960), 1038–1041 (Russian); translated as Soviet Math. Dokl. **1**, 394–397.

The systems considered are of the form

$$(1)\qquad dx/dt = X(x)\quad x=(x_1, x_2, \cdots, x_n),\quad n\geqq 3,$$

where the right members are vector functions whose components possess multiple power series in the x_s, $s=1, 2, \cdots, n$. No terms of zero or first degree are present in these series. The aim of the paper is to construct solutions that tend to zero as $t\to\infty$, or $t\to-\infty$, and to find their asymptotic form for large t. To this end the author constructs a simplified system

$$(2)\qquad dx/dt = \Phi(x),$$

where each component of the vector $\Phi(x)$ is a polynomial formed with certain ones of the terms of low degree in the components of $X(x)$. For the simplified system particular solutions of the form $x_s^*=w_s(c+t)^{-q_s}$, $\bar{x}_s=w_s(c+t)^{-q_s}$, $s=1, 2, \cdots, n$, exist where w_s, $\bar{w}_s$, c, $q_s>0$ are certain constants. The equation (1) is now treated as a perturbation of (2), and the existence of families of solutions that are asymptotically close to x_s^* or $\bar{x}_s$ for $t\to+\infty$ or $t\to-\infty$, respectively, is proved. The number of parameters in these families depends on the real parts of the eigenvalues for the matrix of the variational equation.

W. Wasow (Madison, Wis.)

12284:

Pontryagin, L. S.; Rodygin, L. V. Approximate solution of a system of ordinary differential equations involving a small parameter in the derivatives. Dokl. Akad. Nauk SSSR **131** (1960), 255–258 (Russian); translated as Soviet Math. Dokl. **1**, 237–240.

The authors consider the system

$$\varepsilon \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y), \tag{1}$$

for the k, l dimensional vectors x, y, with $f, g \in C^2(x, y)$ in a certain open region Γ of (x, y) space. The following assumptions are made. (i) $d\bar{x}/dt = f(\bar{x}, y)$ has a unique, stable limit circle for each fixed y in G, the projection of Γ in y-space (restricted to a compact subregion D if necessary). This solution is $x^*(t, y)$, of period $T(y)$. (ii) $d\bar{y}/dt = \bar{g}(\bar{y}) = (1/T(\bar{y}))\int_0^{T(\bar{y})} g(x^*(\tau, \bar{y}), \bar{y})d\tau$ has $\bar{y}(t) \in G$ for $0 \leq t \leq L < \infty$. (iii) $y(t) \in D$, $0 \leq t \leq L$.

By considering an appropriate Lyapunov function, they establish the result that any solution of (1) starting in the domain of attraction of $x^*(t, y)$ in Γ has its x-components performing rapid oscillations about the limit cycle of $\bar{y}(t)$ with approximate period $\varepsilon T(\bar{y}(t))$, while the y-components remain close to $\bar{y}(t)$, uniformly in t, $0 \leq t \leq L$, ($\bar{y}(t)$ having the same initial value as in (1)). The precise statement of the theorem is, "There exists a function $\phi(t, \varepsilon)$ (phase) depending smoothly on t, such that if $\delta > 0$, then for $\delta \leq t \leq L$

$$\left| \varepsilon \frac{d\phi}{dt} - \frac{1}{T(\bar{y}(t))} \right| = O(\varepsilon),$$

$$|x(t, \varepsilon) - X(\phi(t, \varepsilon), \bar{y}(t))| = O(\varepsilon),$$

$$|y(t, \varepsilon) - \bar{y}(t)| = O(\varepsilon),$$

uniformly with respect to t". ($X(\phi, y) = x^*(T(y)\phi, y)$.)

G. Latta (Stanford, Calif.)

12285:

Munteanu, I. Un critère de délimitation pour la solution d'un système non linéaire d'équations différentielles. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 237–243. (Romanian. Russian and French summaries)

Let E_s denote real Euclidean s-space. The author considers a pair of differential equations $dx/dt = A(x) + F(t, x, y)$, $dy/dt = B(y) + G(t, x, y)$, where x, A, F are in E_p and y, B, G are in E_q. It is assumed that positive definite scalar functions $U(x)$, $V(y)$ exist such that $(\text{grad } U, x) \geq hx$, $\|\text{grad } U\| \leq H\|x\|$, $(\text{grad } V, y) \leq -k\|y\|$, and $\|\text{grad } V\| \leq K\|y\|$ for $\|x\| > a$, $\|y\| > b$, where h, H, k, K, a, b are positive constants; it is assumed further that for $\|x\| > a$, $\|y\| > b$, $-\infty < t < \infty$, one has $\|F\| < m\|x\|^{-s}$, $\|G\| < M\|x\|^{-S}$, where m, M, s, S are positive constants. It is then shown that the differential equations have a solution $x(t)$, $y(t)$ which is bounded for $-\infty < t < \infty$. The proof, not all details of which are clear, is based on the method of Wazewski [see L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Springer, Berlin, 1959; MR **22** #9673; pp. 179–182].

W. Kaplan (Ann Arbor, Mich.)

12286:

Yü, Ming-chin. On problems of linear differential equations of H. Poincaré and H. Perron. Advancement in Math. **3** (1957), 427–433. (Chinese)

The author re-establishes the asymptotic behavior of the solutions of linear ordinary differential equations found by Poincaré and Perron as the independent variable tends to ∞ by means of a simple transformation reducing them to the classical case of Bendixson's equations.

S. S. Shu (Lafayette, Ind.)

12287:

Yü, Ming-chin. On the boundedness and stability of solutions of difference-differential equations. Advancement in Math. **4** (1958), 288–296. (Chinese)

The author extends the methods due to Bellman and Sansone to give criteria for the boundedness and stability of the solutions of linear differential-difference equations of first and second order with variable coefficients.

S. S. Shu (Lafayette, Ind.)

12288:

Liu, Yung-ching. Stability of solutions of differential-difference equations. Advancement in Math. **4** (1958), 297–303. (Chinese. English summary)

Algebraic criteria are given for the stability of solutions of the linear differential-difference equation $Au'(t) + Bu(t) + Cu'(t-\delta) + Du(t-\delta) = 0$ and its generalizations.

S. S. Shu (Lafayette, Ind.)

12289:

Sugiyama, Shohei. On the existence and uniqueness theorems of difference-differential equations. Kōdai Math. Sem. Rep. **12** (1960), 179–190.

The author establishes existence and uniqueness theorems for the solutions of the vector difference-differential equations $x'(t) = f(t, x(t), x(t-1))$ and $x'(t) = f(t, x(t), x(t-1), x'(t-1))$ using Tychonoff's fixed-point theorem.

E. Pinney (Berkeley, Calif.)

12290:

Ladyženskaya, O. A. On non-stationary operator equations and their applications to linear problems of mathematical physics. Mat. Sb. (N.S.) **45 (87)** (1958), 123–158. (Russian)

In some earlier papers [notably Dokl. Akad. Nauk SSSR **102** (1955), 207–210; Mat. Sb. (N.S.) **39 (81)** (1956), 491–524; MR **17**, 161; **19**, 279] the author has solved Cauchy's problem for operator equations of the type $u' + Su = f$, $u'' + Su = f$, $S_1u'' + S_2u' + S_3u = f$ where $u = u(t)$ depends on a real variable and has values in a Hilbert space, and the S are unbounded operators depending on t and satisfying certain conditions.

In this paper she reviews these results and points out that they are applicable to a number of boundary problems in cylindrical domains whose generators are parallel to the t-axis. The theory covers parabolic and hyperbolic equations and also, e.g., the time-dependent Schrödinger equation.

There are also a few remarks on the method of finite differences and the continuity method.

L. Gårding (Lund)

12291:

Keller, H. H. Über die Differentialgleichung erster Ordnung in normierten linearen Räumen. Rend. Circ. Mat. Palermo (2) **8** (1959), 117–144.

This paper deals with the general theory of first-order completely integrable differential equations in normed linear spaces. Let X be a normed linear space, let Y be a Banach space, and let $L(X; Y)$ be the Banach space of all linear mappings of X into Y. Let G and H be connected open sets in X and Y, respectively, and consider a function $F: G \times H \to L(X; Y)$. The differential equation in

question is $f'(x)=F(x,f(x))$. Here the solution function f is to be defined on a subset G_1 of G, with values in Y, and f': $G_1 \to L(X;Y)$ is to be the Fréchet derivative of f. The differential equation is called completely integrable if, for each (x_0, y_0) in $G\times H$, there is a solution f defined in a neighborhood of x_0, with the property that $f(x_0)=y_0$.

The condition of complete integrability, and an existence theorem for this equation, in the general Banach space situation, were given by Michal and Elconin in Acta Math. **68** (1937), 71–107. The present paper considers the problem with the notation and techniques exhibited in the recent book, *Absolute analysis*, by F. and R. Nevanlinna [#12176]. The basic assumption about F is that it is uniformly continuously differentiable as a function of (x, y). To explain briefly the contribution of this paper, suppose $x_0=0$, and let G and H be spherical neighborhoods of x_0 and y_0 respectively. If attention is focussed on a 1-dimensional subspace X_1 of X, then the problem, with X_1 in place of X, has a solution f_1, defined near 0, such that $f_1(0)=y_0$. It can be proved that there is a neighborhood G_0 of $0 (G_0 \subset G)$ such that the solution f_1 exists in $X_1 \cap G_0$, no matter how X_1 is chosen. All of these functions f_1 can then be assembled into a function $\bar{f}$ defined in G_0; of course $\bar{f}(0)=y_0$. It turns out to be uniformly continuously differentiable. Now let $R(x, y)$ be the skew-symmetric part of the operator $\partial F(x, y)/\partial x + (\partial F(x, y)/\partial y)F(x, y)$. This operator, for fixed x, y, is a member of $L(X; L(X; Y))$. If $R(x, \bar{f}(x))=0$ in G_0, $\bar{f}$ is a solution of the differential equation in G_0. The necessary and sufficient condition for complete integrability is that $R(x, y)=0$ in $G\times H$. To make clear the definition of R, suppose A is a member of $L(X; L(X; Y))$. The value of A at x_1 is denoted by Ax_1; this is an element of $L(X; Y)$. The value of Ax_1 at x_2 is written Ax_1x_2. If B is the skew-symmetric part of A, B is defined by

$$Bx_1x_2 = \tfrac{1}{2}[Ax_1x_2 - Ax_2x_1].$$

For the case in which F is linear in y, the problem has been treated from a similar point of view, but with special advantage taken of the linearity, in two notes by R. Nevanlinna [C. R. Acad. Sci. Paris **247** (1958), 1953–1954, 2087–2090; MR **21** #2108, 2109].

A. E. Taylor (Los Angeles, Calif.)

12292:

Massera, J. L. Sur l'existence de solutions bornées et périodiques des systèmes quasi-linéaires d'équations différentielles. Ann. Mat. Pura Appl. (4) **51** (1960), 95–105.

Let $t \to A(t)$ be a mapping of $J=[0, \infty)$ into the Banach space of continuous linear operators of a real or complex reflexive Banach space X into itself which is uniformly Bochner integrable in every bounded subinterval of J, and suppose that the set $X_0 \subset X$ of initial points of bounded solutions of (1) $x'+A(t)x=0$ is a closed subspace and has a closed complement. Let h be a weakly continuous mapping of $X\times J$ into X such that $\|h(x, t)\| \leq \varphi(\|x\|)$ where φ is continuous in J and $\varphi(r)=o(r)$ as $t\to\infty$. The author proves that if there exists an exponential dichotomy for the solutions of (1) then (2) $x'+A(t)x=h(x, t)$ has for each $x_0 \in X_0$ at least one bounded solution u such that $P_0u(0)=x_0$, where P_0 is the projection associated with X_0. Moreover, if $A(t+1)=A(t), h(x, t+1)=h(x, t)$ for every $t\in J$, $x\in X$, then (2) has at least one periodic solution with period 1. This extends earlier joint results of the author and Schäffer [Ann. of Math. (2) **67** (1958), 517–573; **69** (1959), 88–104; MR **20** #3466; **21** #756] where h is supposed to be Lipschitzian in $X\times J$ for a sufficiently small constant. It also generalizes a theorem of Demidovič [Mat. Sb. (N.S.) **40** (**82**) (1956), 73–94; MR **18**, 738] for finite-dimensional X in which A is assumed to be constant and to have eigenvalues with non-zero real part (which is equivalent to exponential dichotomy). The present result on the existence of a bounded solution of (2) is very similar to a theorem of Corduneanu for the case $X=R^n$ [An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 33–36; MR **22** #5766].

H. A. Antosiewicz (Los Angeles, Calif.)

12293:

Massera, J. L.; Schäffer, J. J. Linear differential equations and functional analysis. Bol. Soc. Mat. Mexicana (2) **5** (1960), 42–48.

This is a report (without proofs) on partially completed and published research by the authors [Ann. of Math. (2) **67** (1958), 517–573; **69** (1959), 88–104, 535–574; Math. Ann. **139** (1960), 287–342; MR **20** #3466; **21** #756, 3638; **22** #8181] on the application of functional analysis to linear (and some non-linear) ordinary differential equations. The equations considered are essentially of the form $\dot{x}(t)+A(t)x(t)=f(t)$ where x and f are elements of Banach function spaces D and B, the central problem being to establish, given f in B, the existence of a solution in D. Among the related papers are those of Kreĭn [Uspehi Mat. Nauk **3** (1948), no. 3 (25), 166–169; MR **10**, 128] and Kučer [Dokl. Akad. Nauk SSSR **69** (1949), 603–606; MR **11**, 360].

H. H. Schaefer (Ann Arbor, Mich.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 12215, 12216, 12268, 12290, B12723.

12294:

Gel'fand, I. M. Some questions of analysis and differential equations. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 3–19. (Russian)

L'auteur énonce quelques problèmes qu'il a trouvé en rédigeant les rapports faits à la Société de Mathématiques de Moscou (1957) et au Congrès international des mathématiciens, tenu à Edinbourgh en 1958. Ces problèmes proviennent de plusieurs branches de mathématique, comme les équations différentielles aux dérivées partielles (divers types d'ellipticité, problèmes correctes, comportement asymptotique, équations quasi-linéaires), la théorie des représentations et la théorie des mesures dans les espaces vectoriels.

Nous reproduisons seulement quelques-uns de ces problèmes, dont l'énoncé est plus simple et ne suppose pas une préparation plus détaillée:

(1) Décrire les composantes connexes de l'ensemble des systèmes hypoelliptiques et spécialement caractériser algébriquement la composante qui contient le système $\Delta_t u=0$.

(2) Décrire les problèmes correctes pour les systèmes $\partial u/\partial t = L(\partial/\partial x_t, x)u$; dans le cas des coefficients constants, peut-on affirmer qu'un problème est correcte si et seulement s'il est correcte en chaque point?

(3) Soient Φ un espace nucléaire et Φ' son dual; construire toutes les mesures sur Φ', qui sont quasi-

invariantes par rapport aux translations avec les éléments de Φ.

L'auteur indique aussi les liaisons entre ce dernier problème, la mécanique quantique et les anneaux de type II. *G. Marinescu* (Bucharest)

12295:

Ghelfand, I. M. Some questions of analysis and differential equations. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **14** (1960), no. 1 (32), 3–19. (Romanian)

Translation of #12294.

12296:

Rasulov, M. L. Residue method for solution of boundary and mixed problems for differential equations. Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Teh. Him. Nauk **1958**, no. 1, 3–12. (Russian. Azerbaijani summary)

This paper is the third part of an article, the first two parts of which appeared in the same Izv. **1957**, no. 12, 3–16. In this part the author applies his methods to problems in hydromechanics and to a problem in heat conduction. *W. S. Loud* (Madison, Wis.)

12297:

Boigelot, A. Méthodes spectrales dans les problèmes aux limites de la physique mathématique. Mém. Soc. Roy. Sci. Liège (5) **2**, no. 1, 1–99 (1959).

Let A be a self-adjoint realization in $H=L^2(V)$, (V open in R^n), of an elliptic self-adjoint differential operator $a=a(x,\ \partial/\partial x)$ with smooth coefficients. Put $B=\int b(\lambda)\,dE_\lambda$ where $A=\int \lambda\,dE_\lambda$ is the spectral resolution and b is a Borel function. By Schwartz's theorem on kernels, $Bf(x)=\int K(x,\ y)f(y)\,dy$ where the kernel K is a distribution. Since K is smooth when b has compact support, the smoothness properties of K depend on the size of b for large λ. This theme is developed at great length in the special case when A is Laplace's operator Δ or a power of Δ. *L. Gårding* (Lund)

12298:

Roseau, Maurice. Recherches sur la théorie des ondes liquides de gravité en profondeur variable (suite). J. Math. Pures Appl. (9) **38** (1959), 275–299.

The author continues his study of the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)\phi(x,\ y)\ =\ 0$$

in a sector, with the boundary conditions

$$\frac{\partial\phi}{\partial y}-\phi\ =\ 0\quad\text{on}\quad y\ =\ 0,$$

$$\sin\alpha\cdot\frac{\partial\phi}{\partial x}+\cos\alpha\cdot\frac{\partial\phi}{\partial y}\ =\ 0\quad\text{on}\quad x\sin\alpha+y\cos\alpha\ =\ 0,$$

begun on same J. **38** (1959), 25–59 [MR **21** #4676]. Suppose that $k>1$, and write $\cos\theta=1/k$; denote by $\phi_1^{(0)}$ and $\phi_2^{(0)}$ the solutions previously denoted by ϕ_1 and ϕ_2. The following results are obtained. (1) Suppose that $\pi-2\theta\neq 2n\alpha$, where n is an integer: (a) if $\alpha=\pi/2m$ where m is an integer, then $\phi_2^{(0)}=0$, and $\phi_1^{(0)}$ has a singularity of order $r^{-\pi/\alpha}$ at the origin; (b) if $3\pi-2\theta=(2n+1)\alpha$ where n is an integer, then $\cos\ (\pi^2/\alpha)\cdot\phi_2^{(0)}-\sin\ (\pi^2/\alpha)\cdot\phi_1^{(0)}=0$, and $\phi_1^{(0)}$ and $\phi_2^{(0)}$ have singularities of order $r^{-\pi/\alpha}$ at the origin; (c) if $\alpha\neq\pi/2m$ and $3\pi-2\theta\neq(2n+1)\alpha$, then the combination $\cos\ (\pi^2/\alpha)\cdot\phi_2^{(0)}-\sin\ (\pi^2/\alpha)\phi_1^{(0)}$ is not zero and has a logarithmic singularity at the origin. (2) It is possible to construct other solutions $\phi_1^{(p)}$, $\phi_2^{(p)}$ vanishing at infinity and having at the origin a singularity of order $r^{-p\pi/\alpha}$. *F. Ursell* (Cambridge, Mass.)

12299:

Étienne, J.; Thunus, J. Fonctions de Green de l'opérateur métaharmonique pour les problèmes de Dirichlet et de Neumann posés dans un intervalle. Bull. Soc. Roy. Sci. Liège **28** (1959), 198–206.

Mediante un metodo già utilizzato da uno degli Autori [lo stesso Bull. **27** (1958), 142–156, 189–200; MR **21** #198] in casi analoghi, gli autori ottengono diverse forme della funzione di Green per i problemi di Dirichlet e di Neumann per l'operatore $-\Delta+k^2$, $k>0$, nell'intervallo $I=\{0\leq x_i\leq a_i,\ i=1,\ \cdots,\ n\}$ dello spazio R^n. La forma principale ottenuta é la seguente

$$G(P,\ P_0)\ =\ \frac{1}{\prod_{i=1}^{n} a_i}\,\frac{1}{2^n}$$

$$\times\ \int_0^\infty \prod_{i=1}^{n}\left\{\theta\left(\frac{x_i-x_i^0}{2a_i},\ \frac{i\pi t}{a_i^2}\right)\mp\theta\left(\frac{x_i+x_i^0}{2a_i},\ \frac{i\pi t}{a_i^2}\right)\right\}l^{-k^2t}\,dt$$

dove $P=(x_1,\ \cdots,\ x_n)$, $P_0=(x_1^0,\ \cdots,\ x_n^0)$, θ é la funzione di Jacobi e i segni $-$ e $+$ corrispondono rispettivamente ai problemi di Dirichlet e di Neumann. *E. Magenes* (Pavia)

12300:

Solonnikov, V. On estimates of Green's tensors for certain boundary problems. Dokl. Akad. Nauk SSSR **130** (1960), 988–991 (Russian); translated as Soviet Math. Dokl. **1**, 128–131.

Viene considerato il sistema: $\Delta\mathbf{v}=\text{grad}\ p+\mathbf{f}$, $\text{div}\ \mathbf{v}=0$ in una regione limitata Ω dello spazio tridimensionale, con frontiera S. La condizione al contorno è $\mathbf{v}|_S=0$. L'autore riprende la costruzione data da Odqvist per il tensore di Green [Math. Z. **32** (1930), 329–375] raffinando e generalizzando alcune maggiorazioni. Egli perviene così a maggiorazioni di tipo Hölderiano per $\partial v/\partial x_k$, $\partial^2 v/\partial x_i\partial x_k$, $\partial p/\partial x_i$.

L'autore enuncia poi questo importante risultato: se $S\in C^2$ vale la limitazione $\|\mathbf{v}\|_{W_p^2(\Omega)}\leq k\|\mathbf{f}\|_{L_p(\Omega)}$. Di questo risultato non è data alcuna dimostrazione; non si vede neppure come esso possa essere dedotto in modo diretto dalle valutazioni per il tensore di Green enunciate prima.

L'autore dà rappresentazioni per le soluzioni del problema: $\text{rot}\ \mathbf{H}=\mathbf{j}$, $\text{div}\ \mathbf{H}=0$, $\mathbf{H}_n|_S=0$ e del problema $\text{rot}\ E=\mathbf{a}$, $\text{div}\ E=0$, $E_\tau|_S=0$ (qui H_n indica la componente di H normale ad S, E_τ la componente tangenziale di E). Per le soluzioni di questo problema vengono enunciate valutazioni di tipo Hölderiano analoghe a quelle date per il problema prima considerato. *G. Prodi* (Trieste)

12301:

Lasota, A. Sur l'existence et l'unicité des solutions d'un problème de Mlle Z. Szmydt relatif à l'équation de la

corde vibrante en fonction de la position du point initial. Ann. Polon. Math. **9** (1960/61), 49–53.

The author proves in the first place that the equation $u_{xy}=0$ has in the rectangle Π defined by $|x|\leq A$, $|y|\leq B$ a uniquely determined solution satisfying the conditions $u_x(x, a)=ku(x, a)$, $u_y(b, y)=lu(b, y)$, $u(x_0, y_0)=u_0$, where a, b, k, l and u_0 are given constants, $|a|\leq B$, $|b|\leq A$, if the point x_0, y_0 is not on the curve $e^{k|x-b|}+e^{l|y-a|}=1$. For $|k|\leq K$, $|l|\leq L$ these curves cover the entire plane except the point-set M given by

$$\min_{|\xi|\leq A} e^{-K|\xi-x|}+\min_{|\eta|\leq B} e^{-L|\eta-y|}>1.$$

Further, it is proved that the equation $u_{xy}=F(x, y)$ has a uniquely determined solution in Π, satisfying $u_x(x, \alpha(x))=G(x, u(x, \alpha))$, $u_y(\beta(y), y)=H(y, u(\beta, y))$, $u(x_0, y_0)=u_0$, where $F(x, y)$, $G(x, u)$, $H(y, u)$, $\alpha(x)$ and $\beta(y)$ are given continuous functions in Π, $(|\alpha(x)|\leq B,\ |\beta(y)|\leq A)$, if $P(x_0y_0)$ is a point of M. *H. Bremekamp* (Delft)

12302:

Guglielmino, Francesco. Su un problema al contorno per le equazioni paraboliche del tipo del calore in due variabili. Ricerche Mat. **9** (1960), 275–310.

The author proves the existence and uniqueness of the solution of boundary-value problems of the second kind for the parabolic equation

$$a(x, y)\frac{\partial^2 u}{\partial x^2}+b(x, y)\frac{\partial u}{\partial x}+c(x, y)u-\frac{\partial u}{\partial y}=f(x, y)$$

in two independent variables, subject to the boundary conditions $u_x(0, y)=\phi(y)$, $u_x(X, y)=\psi(y)$ and the initial condition $u(x, 0)=\chi(x)$, under the following assumptions: (1) In a rectangle R: $0\leq x\leq X$, $0\leq y\leq Y$, the coefficients c and f are square-integrable, a and b are measurable and bounded, and $a\geq\varepsilon>0$. (2) ϕ and ψ are Hölder continuous in $0\leq y\leq Y$ with exponent $>\frac{1}{2}$. (3) χ is absolutely continuous, χ' square-integrable in $0\leq x\leq X$. (4) For almost all y, u_x is absolutely continuous in x. (5) u_x, u_y, u_{xx} are square-integrable in R.

H. C. Kranzer (Garden City, N.Y.)

12303:

Manohar, R. Pointwise bounds for the solutions of certain boundary value problems. J. Indian Math. Soc. (N.S.) **23** (1959), 33–43 (1960).

The author deals with the following problems: (I) $\Delta\Delta w=p(x, y)$ in the region A, and $w=0$, $\partial w/\partial n=0$ on B; (II) $\Delta\Delta R=0$ in A and $R=-r^2\lg r$, $\partial R/\partial n=-\partial(r^2\lg r)/\partial n$ on B; (III) $\Delta\Delta q=p(x, y)$ in A and $q=-\varepsilon r\lg r$, $\partial q/\partial n=-\varepsilon\partial(r^2\lg r)/\partial n$ on B. Here $\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$, A is a plane domain with boundary B. R is a regular function associated with the Green's function $g(x, y; x_0, y_0)=r^2\lg r+R(x, y)$, ε is any real parameter. In § 1 the inequalities obtained by Diaz and Greenberg [J. Math. and Phys. **27** (1948), 193–201; MR **10**, 213] for the first biharmonic problem are modified. In § 2 the method of hypercircle [cf. J. L. Synge, *The hypercircle method in mathematical physics: a method for the approximate solution of boundary value problems*, Cambridge Univ. Press, New York, 1957; MR **20** #4073] is applied to the problems of § 1. The same procedure can be applied to the problems of Dirichlet and Neumann. The results hold in n-dimensions. *M. Pinl* (Cologne)

12304:

Aeppli, Alfred. On the uniqueness of compact solutions for certain elliptic differential equations. Proc. Amer. Math. Soc. **11** (1960), 826–832.

Let O be a fixed point in R^{n+1}, $r=r(p)$ be the distance of the point p from O, and H_ν the νth mean curvature of an n-dimensional surface in R^{n+1}. It is first shown that the only compact n-surfaces of class C^2 without boundary for which $r^\nu H_\nu=1$ are spheres with centers at O. The same result is obtained if $r^\nu H_\nu$ is replaced by $\sum a_\nu r^\nu H_\nu=\sum a_\nu>0$, where each $a_\nu(p)\geq 0$ and is of class C' on $R^{n+1}-O$, provided the surface is assumed simple (i.e., to have no self-intersections) and to contain O in its interior. It is next shown that any simple compact solution surface with O in its interior, of an equation of the form $r(p)H_1(p)=f(p)$, where f is homogeneous of degree 0 if we take the origin at O, is star-shaped with respect to O and any two such surfaces are homothetic. The same result is obtained for an equation of the form $\sum_\nu a_\nu(p)r^\nu(p)H_\nu(p)=f(p)>0$, where f and the a_ν are homogeneous of degree 0, provided only ovaloid surfaces are admitted. The author added in proof the remark that A. D. Aleksandrov obtained similar results in a paper in Vestnik Leningrad. Univ. **13** (1958), no. 7, 14–26 [MR **21** #907]. *C. B. Morrey, Jr.* (Berkeley, Calif.)

12305:

Pliš, A. Non-uniqueness in Cauchy's problem for differential equations of elliptic type. J. Math. Mech. **9** (1960), 557–562.

E. Holmgren has shown in 1901 that Cauchy's problem for linear partial differential equations with analytic coefficients and with initial data given on a non-characteristic manifold has at most one smooth solution. T. Carleman [Ark. Math. Astr. Fys. **26B** (1939), no. 17; MR **1**, 55] and A. P. Calderón [Amer. J. Math. **80** (1958), 16–36; MR **21** #3675], the former for the case of two independent variables, extended this theorem to systems with non-analytic coefficients under the additional assumption that the characteristics are non-multiple. Examples by the author [Bull. Acad. Pol. Sci. Sér. Sci. Math. Astr. Phys. **2** (1954), 55–57; MR **16**, 258] and by E. de Giorgi [Rend. Mat. e Appl. (5) **14** (1955), 382–387; MR **16**, 1119] show that multiplicity of real characteristics may cause non-uniqueness, even in the case of C^∞-coefficients. It is the question whether the multiplicity of essentially complex characteristics can be admitted in the uniqueness theorem. L. Hörmander proved a uniqueness theorem for equations having constant leading coefficients with at most double multiplicity of complex characteristics. In the present paper the author gives a remarkable example of a system consisting of $4(m+3)$ real equations for $4(m+3)$ real functions $(m\geq 0)$ of two variables with constant principal part and C^m-coefficients, whose (imaginary) characteristics are $(m+3)$-fold, and for which uniqueness of Cauchy's problem does not hold.

J. C. C. Nitsche (Rio Piedras)

12306:

Hörmander, Lars. On the uniqueness of the Cauchy problem. II. Math. Scand. **7** (1959), 177–190.

Consider the differential inequality

$$(*)\qquad |P(x, D)u|\leq K\sum_{|\alpha|<m}|D^\alpha u|,$$

in which $P(x, D)$ denotes a linear, elliptic partial differential operator of order m with Lipschitz continuous coefficients. One of the main results of this paper, extending in important respects those of A. P. Calderón [Amer. J. Math. **80** (1958), 16–36; MR **21** #3675], is that for homogeneous P with simple (complex) characteristics, if a C^m solution u of (*) vanishes in the part of a neighborhood of 0 in which $(x^1) < (x^2)^2 + \cdots + (x^n)^2$, then u vanishes in a full neighborhood of 0. Analogous statements are made for homogeneous P with non-singular characteristics and also for products $P = P_1P_2$ of two operators of the preceding two types. The proofs of these statements are outgrowths of methods previously developed by the author [Math. Scand. **6** (1958), 213–225; MR **21** #3674] in the constant coefficient case and are based on general integral inequalities of which the following, for homogeneous P with simple characteristics, is typical. Let $\phi_\delta = (x^1 - \delta)^2 + \delta((x^2)^2 + \cdots + (x^n)^2)$. The inequality states that, if u is in C^∞ and is zero outside a specified neighborhood of 0, then for $|\alpha| \leq m$,

$$(1+\delta^2\tau)^{m-|\alpha|-1}\tau^{m-|\alpha|}\int |D^\alpha u|^2 e^{2\tau\phi_\delta}\,dx \leqq C\int |P(x, D)u|^2 e^{2\tau\phi_\delta}\,dx,$$

provided that $\tau\delta > M$ and $\delta < \delta_0$, where M and δ_0 are constants. This inequality is a consequence, in part, of a generalization by the author of a prior inequality of F. Trèves [Acta Math. **101** (1959), 1–139].

A. Douglis (College Park, Md.)

12307:

Mihaĭlov, V. P. A solution of a mixed problem for a parabolic system using the method of potentials. Dokl. Akad. Nauk SSSR **132** (1960), 291–294 (Russian); translated as Soviet Math. Dokl. **1**, 556–560.

Let Ω be a domain bounded by $t = 0$, $t = T$ and a surface Γ, where Γ satisfies Lyapunov's condition. Consider in Ω a non-homogeneous parabolic system in the sense of Petrowski and mixed boundary conditions: $u = 0$ on $t = 0$, $\partial^i u/\partial n^i = 0$ on Γ, where n is the normal to Γ and $0 \leq i \leq p-1$ ($2p$ the order of the system). The coefficients of the system are assumed to be sufficiently smooth so that a fundamental matrix exists. The author then asserts that Green's matrix can be constructed for the mixed problem by the parametric method. The construction depends upon the theory of integral equations, but the crucial steps are (1) the derivation of the jump relations for the parabolic potentials based on the given fundamental matrix, and (2) writing these relations in a suitable way (employing fractional derivatives) so that the Fredholm theory can be applied. Full statements concerning (1) and (2) are given, but no proofs.

A. Friedman (Minneapolis, Minn.)

12308a:

Nagumo, Mitio. On singular perturbation of linear partial differential equations with constant coefficients. I. Proc. Japan Acad. **35** (1959), 449–454.

12308b:

Kumano-go, Hitoshi. On singular perturbation of linear partial differential equations with constant coefficients. II. Proc. Japan Acad. **35** (1959), 541–546.

Let

$$L_\varepsilon = \sum_{j=1}^{l} P_j(D, \varepsilon)(\partial/\partial t)^j,$$

where $D = (\partial/\partial x_1, \cdots, \partial/\partial x_m)$ and the $P_j(\xi, \varepsilon)$ are matrices of polynomials in $\xi = (\xi_1, \cdots, \xi_m)$ with constant coefficients which depend continuously on the parameter $\varepsilon \geqq 0$. Assume $P_l(\xi, \varepsilon) = P_l(\varepsilon)$ and $\det P_l(\varepsilon) \neq 0$ for $\varepsilon > 0$. These papers are concerned with the relationship between solutions of $L_\varepsilon[u] = f(t, x, \varepsilon)$ for $(t, x) \in (0, \infty) \times E^m$ and $L_0[u] = f(t, x, 0)$ satisfying the same initial conditions on the hyperplane $t = 0$, especially the case $\det P_l(0) = 0$.

Let H_p be the completion of functions $\varphi \in C_0^\infty(E^m)$ with respect to the norm $\|\varphi\|_p^2 = \int (1 + |\xi|^2)^p |\hat{\varphi}(\xi)|^2 d\xi$, where $\hat{\varphi}$ is the Fourier transform of φ and $p \gtrless 0$. A function $u = u(t)$ defined for $t \in J$ with values in H_p is said to be a generalized H_p-solution of $L_\varepsilon[u] = f$ if u is the limit in the H_p-topology, uniformly on compact subsets of J, of a sequence $\{u_\nu(t)\}$ which is such that $L_\varepsilon[u_\nu] \to f$ in H_p.

In part I the author gives a necessary and sufficient condition for H_p-stability of generalized H_p-solutions u_ε of $L_\varepsilon[u] = f(t, x, \varepsilon)$ with respect to an H_p-solution u_0 of $L_0[u] = f(t, x, 0)$. u_ε is H_p-stable with respect to u_0 if $u_\varepsilon \to u_0$ in H_p uniformly for $t \in [0, T]$ whenever $f(t, x, \varepsilon) \to f(t, x, 0)$. The stability condition is in the form of inequalities which must be satisfied by the solution of $\sum_{\mu=0}^{l} P_\mu(i\xi, \varepsilon)(d/dt)^\mu y = 0$; $(d/dt)^{k-1} y_j(0) = \delta_{kj}$ for $k = 1, \cdots, l$.

In part II, the results of part I are applied to the equation $\varepsilon(\partial/\partial t)^2 u + a(\partial/\partial t)u + Q(D)u = f(t, x, \varepsilon)$. The author deduces conditions on the complex constant a and on the structure of the polynomial $Q(i\xi)$ which guarantee H_p-stability.

D. G. Aronson (Minneapolis, Minn.)

12309:

Leray, Jean. La solution unitaire d'un opérateur différentiel linéaire (Problème de Cauchy. II). Bull. Soc. Math. France **86** (1958), 75–96.

[For part I, see same Bull. **85** (1957), 389–429; MR **21** #2102.] This article gives details of the author's note in C. R. Acad. Sci. Paris **245** (1957), 2146–2152 [MR **20** #158].

S. Łojasiewicz (Kraków)

12310:

Peetre, Jaak. Opérateurs différentiels quasi linéaires hypoelliptiques. C. R. Acad. Sci. Paris **248** (1959), 3401–3403.

Considérons un opérateur P de la forme

$$P = \sum_j a^j(x, N_{j1}, \cdots, N_{jk_j})M_j + a^0(x, N_{01}, \cdots, N_{0k_0}),$$

où a^0, a^1, $\cdots$ sont des fonctions indéfiniment différentiables sur $\Omega \times C^{k_0}$, $\Omega \times C^{k_1}$, $\cdots$ (Ω étant un ouvert dans R_x^n), et $M_1, \cdots, N_{01}, \cdots, N_{0k_0}, N_{11}, \cdots, N_{1k_1}, N_{21}, \cdots$ sont des opérateurs différentiels à coéfficients constants.

Soit maintenant f, une fonction suffisamment différentiable, solution de $Pf = 0$. On suppose que l'opérateur différentiel linéaire

$$P_0(x, D) = \sum_j a^j(x, N_{j1}f, \cdots, N_{jk_j}f)M_j$$

soit formellement hypoelliptique, c'est-à-dire qu'on suppose les conditions suivantes (a) et (b):

(a) Pour tout $x \in \Omega$, on a

$$\sum_j |\hat{M}_j(\xi)|/|P_0(x, \xi)| = O(1), \qquad \xi \to \infty,$$

où $P_0(x, \xi)=\sum_j a^j \hat{M}_j(\xi)$, $\hat{M}_j(\xi)$ étant la transformée de Fourier de M_j:
(b) Il existe $\delta>0$ tel que, pour toute $\alpha\neq 0$,

$$\sum_j |\hat{M}_j^{(\alpha)}(\xi)| / \sum_j |\hat{M}_j(\xi)| = O(|\xi|^{-\delta}), \qquad \xi \to \infty,$$

où $\hat{M}_j^{(\alpha)} = (\partial/\partial\xi)^\alpha \hat{M}_j(\xi)$.
On suppose de plus (condition quasi-linéaire):
(c) Il existe un nombre $\tau>0$ tel que

$$\sum |\hat{N}_{jk}(\xi)| / \sum_j |\hat{M}_j(\xi)| = O(|\xi|^{-\tau}), \qquad \xi \to \infty.$$

Sous les hypothèses ci-dessus, f est une fonction indéfiniment différentiable.

La démonstration est esquissée. On trouve la démonstration détaillée dans sa thèse [Lund, 1959].

S. Mizohata (New York)

12311:

Kružkov, S. N. The Cauchy problem in the large for certain non-linear first order differential equations. Dokl. Akad. Nauk SSSR **132** (1960), 36–39 (Russian); translated as Soviet Math. Dokl. **1**, 474–477.

The author considers two problems: Cauchy's problem for the equation

$$(1) \quad u_t+\phi(u_x) = 0, \quad \phi''(v) \geqq a > 0, \quad \phi(0) = \phi'(0) = 0$$

with the initial condition

$$(2) \qquad u(0, x) = u_0(x)$$

($u_0(x)$ is an arbitrary bounded function), and Cauchy's problem for the equation

$$(3) \qquad v_t+[\phi(v)]_x = 0, \quad \phi''(v) \geqq a > 0, \quad \phi'(0) = 0,$$

if the initial conditions can be identified with certain functionals.

For the problems (1), (2) the author defines the general solution and presents theorems about its existence, uniqueness, continuous dependence on initial conditions and about some characteristic properties of this solution, namely, its continuity for $t>0$ (independently of the continuity or discontinuity of the initial function $u_0(x)$).

He first discusses the case when $u_0(x)$ satisfies Lipschitz's condition on the entire x-axis (with the same constant K). In this case, as he shows, the solution is $u(t, x) = \min_{-\infty<\sigma<\infty} I(\sigma, t, x)$, where

$$(4) \qquad I(\sigma, t, x) = u(x-t\phi'(\sigma))+t\int_0^\sigma \xi\phi''(\xi)\,d\xi.$$

The continuous dependence on initial conditions is proved with the help of properties of the solutions of equations

$$(5) \qquad u_t+\phi(u_x) = \varepsilon u_{xx}, \qquad \phi(0) = 0, \qquad \varepsilon > 0,$$

with the same initial condition. The relation between solutions of (5) and of (1) and (2) is discussed.

In the case of the discontinuous initial condition $u_0(x)=\mu(x)$, the generalised solution of (1) and (2) (the definition of which is rather modified in comparison with the case of continuous $u_0(x)$) is determined also by the formulae (4). A theorem is presented about convergence of $u_n(t, x)$, corresponding to continuous initial functions $u_n(x)$, if $u_n(x)\to\mu(x)$.

The work ends with the definition of the problem corresponding to the equation (3) and by presenting some notes about existence, uniqueness and stability (with respect to the initial conditions) of the solution.

Detailed proofs of the results of the work will be published later.

K. Rektorys (Prague)

12312:

Gol'denveizer, A. L. Asymptotic integration of linear partial differential equations with a small principal part. J. Appl. Math. Mech. **23** (1959), 44–74 (35–57 Prikl. Mat. Meh.).

This article is a continuation of the author's work in Prikl. Mat. Meh. **22** (1958), 657–672 [MR **22** #2802] which is quoted extensively. Let $L(\phi)$ and $N(\phi)$ be linear differential expressions of order l and n, respectively ($n>l$), in the two independent variables α and β, and let h be a small positive parameter. The aim of the paper is to construct series expansions for certain solutions of the differential equation $hN(\Phi)+L(\Phi)=0$. Assuming n to be even, these solutions are to satisfy for $\alpha=\alpha_0>0$ conditions of the form $\partial^\mu\Phi/\partial\alpha^\mu = k^\mu g^{(\mu)} e^{ik\phi}$, $\mu=0, 1, 2, \cdots, \frac{1}{2}n-1$, where k is a large parameter and $g^{(\mu)}$ and ϕ are given functions of β. The parameters h and k are assumed to be linked by a relation of the form $k=h^{-t}$, where t is a certain positive rational number.

In the first part of the paper the author constructs expressions involving series in negative powers of k which satisfy the differential equation formally. These expressions are of the form $\Phi=\Phi_* e^{kf}$, where Φ_* is a series in descending fractional powers of k, and f is a polynomial in fractional powers of k^{-1} with coefficients depending on α and β. The question as to the asymptotic character of the series Φ_* is discussed but not completely solved. The recursive scheme by means of which the coefficients of f and Φ_* are calculated depends decisively on whether t is greater than, equal to, or smaller than $1/(n-l)$. In every case this calculation involves the solution of sequences of certain first-order partial differential equations.

Particular solutions that satisfy the imposed boundary conditions are found as a finite sum of solutions of the form just described. This involves calculations too involved to be described here. Certain additional assumptions have to be introduced. For instance, when $t>1/(n-l)$, the operator N has to be elliptic; when $t<1/(n-l)$, L must be elliptic.

The method of this paper is an extension of the author's work in his monograph on the theory of elastic shells [*Teoriya uprugih tonkih obolochek*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 645].

W. Wasow (Madison, Wis.)

12313:

Leontovič, N. M. The asymptotic expansion of boundary value problems for partial differential equations. Dokl. Akad. Nauk SSSR **131** (1960), 742–745 (Russian); translated as Soviet Math. Dokl. **1**, 326–329.

The boundary value problem considered is $L_\varepsilon \equiv \varepsilon^2\Delta\Delta u + L_2 u = h$ in Q, $u|_\Gamma = 0, \Delta u|_\Gamma = 0$. Here Q is a bounded region with boundary Γ of n-dimensional space, $h=h(x)$ is a function of the position vector $x=(x_1, x_2, \cdots, x_n)$, and L_2 is an elliptic operator of the form

$$L_2 \equiv -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

$$c - \frac{1}{2}\sum_{i=1}^n \frac{\partial b_i}{\partial x_i} \geqq \alpha^2 > 0.$$

ε is a small positive parameter. It is shown that the solution u has the form $u=\tilde{u}_N+\varepsilon^2\tilde{v}_N+w$, where $\tilde{u}_N=\sum_{j=0}^{N}\varepsilon^j u_j$, $\tilde{v}_N=\sum_{j=0}^{N}\varepsilon^j v_j$, and the $\mathscr{L}_2$-norm of w is $O(\varepsilon^{N+1})$. The integer N is arbitrary, provided the coefficients of the differential equation are sufficiently regular. The u_j depend on x, but not on ε. The function $v_j=v_j(x,\varepsilon)$ are of the "boundary layer" type, i.e., they tend exponentially to zero with ε except in a strip along the boundary of width $O(\varepsilon)$. The function u_0 is the solution of the "degenerate problem" $L_2u_0=h$, $u|_\Gamma=0$. It is shown, further, that the $\mathscr{L}_2$-norms of the derivatives of order 1, 2, 3, 4 of the error term w have, respectively, the orders $O(\varepsilon^{N+1})$, $O(\varepsilon^N)$, $O(\varepsilon^{N-1})$, $O(\varepsilon^{N-1})$. The author states, without giving details, that strictly analogous results hold for the problem with boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma=0,\quad \left.\frac{\partial}{\partial n}(\Delta u)\right|_\Gamma=0,$$

provided $\{a_{ij}(x)\}$ is the identity matrix. The proof follows the method developed by M. I. Višik and L. A. Lyusternik in Uspehi Mat. Nauk **12** (1957), no. 5 (77), 3–122 [MR **20** #2539]. *W. Wasow* (Madison, Wis.)

POTENTIAL THEORY

See also 12220, 12221, 12250, 12299, 12303.

12314:

Bach, W. On the asymptotic behaviour of harmonic functions in the semi-space. Ann. Polon. Math. **9** (1960/61), 137–144.

Let $f(x, y)$ be continuous and bounded in the xy-plane and set

$$u(x,y,z)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\frac{dG}{dn}\,d\xi\,d\eta,$$

where dG/dn is the inner normal derivative for G, the Green's function for half space $z>0$. Then $u(x, y, z)$ is a solution of the Dirichlet problem in $z>0$ with $f(x, y)$ as the boundary function. Next, let L denote the half-line $x=x_0+at$, $y=y_0+bt$, $z=z_0+ct$, $t\in\langle 0,\infty)$, where a and b are real, c positive and (x, y, z) a point in $z\geq 0$.

The author first proves that the upper and lower limits of $u(x, y, z)$, as $(x, y, z)\to\infty$ along L, depend only on a, b, c and not on the point (x, y, z). An example is given to show that these limits need not be equal. Next the author shows that if the limit $\lim_{r\to\infty}\int_0^{2\pi}f(r\cos\phi, r\sin\phi)d\phi$ exists, then $\lim_{z\to\infty}2\pi u(x, y, z)$ exists and both limits are equal. The paper is concluded with a pair of related results. *F. W. Gehring* (Ann Arbor, Mich.)

12315:

Stein, Elias M.; Weiss, Guido. On the theory of harmonic functions of several variables. I. The theory of H^p-spaces. Acta Math. **103** (1960), 25–62.

L'objet essentiel de ce travail est d'étendre le résultat classique selon lequel toute fonction $F(x+iy)$ analytique dans le demi-plan $y>0$ et satisfaisant à la condition $\int_{-\infty}^{+\infty}|F(x+iy)|^p dx\leq A<\infty$ pour un $p>0$ et tout $y>0$, admet, lorsque $y\to 0$, des valeurs limites $F(x)$, à la fois presque partout et en norme.

On se place dans le demi-espace E_{n+1}^+ des points (X, y), où $y>0$ et $X=(x_1, \cdots, x_n)$ est dans l'hyperplan E_n, d'équation $y=0$, qui limite E_{n+1}^+; $n+1$ fonctions harmoniques dans E_{n+1}^+ sont dites former un système conjugué: $F(X, y)=(u_1(X, y), \cdots, u_{n+1}(X, y))$, si elles sont les composantes du gradient d'une certaine fonction harmonique dans E_{n+1}^+.

Alors, si l'on pose $|F|=(\sum_1^{n+1}u_i^2)^{1/2}$, $|F|^r$ est sous-harmonique pour tout $r\geq(n-1)/n$; si de plus

$$\int_{E_n}|F(X,y)|^p\,dX\leq A<\infty$$

pour un $p\geq(n-1)/n$ et tout $y>0$ (ce qui définit la "classe H^p"), $|F|^{(n-1)/n}$ admet des majorantes harmoniques dans E_{n+1}^+, et la plus petite est l'intégrale de Poisson-Stieltjes d'une fonction $f\in L^p(E_n)$ ou d'une mesure finie sur E_n selon que $p>(n-1)/n$ ou $p=(n-1)/n$. Cela suffit pour prouver l'extension en vue: toute $F(X, y)\in H^p$, $p\geq(n-1)/n$ admet presque partout sur E_n une limite lorsque $y\to 0$; si $p>(n-1)/n$, cette limite est aussi limite en norme.

Applications: généralisation d'un théorème de F. et M. Riesz sur propriétés de certaines intégrales fractionnaires sur les espaces H^p. *L. Naïm* (Grenoble)

12316:

Ninomiya, Nobuyuki. Sur le principe du maximum et le balayage. Japan. J. Math. **29** (1959), 68–77.

The author studies the 'inverse' principles of potential theory for a continuous positive symmetric kernel K on a locally compact space Ω, by the methods he used to study the 'direct' or 'ordinary' principles in J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 147–174 [MR **21** #776]. The first [resp. second] principle of the minimum is said to hold if, for every positive measure μ with compact support F, the inequality $\int K(x, y)\mu(dy)\geq 1$ [resp. $\geq K(x, p)$, with p outside F] for x in F implies the same inequality for all x in Ω. For K bounded, the author proves that the first [resp. second] principle of the minimum is equivalent to the principle of inverse equilibrium [resp. inverse balayage]; that the potential resulting from inverse balayage on a compact set is unique; that the potential of inverse equilibrium for a compact set is unique, provided the principle of inverse equilibrium holds; and that either principle of the minimum implies that K is of negative type, that is, $\|\mu\|\cdot\|\nu\|\leq\langle\mu,\nu\rangle$ if μ and ν are positive measures. The weak principle of balayage (equilibrium) implies the corresponding ordinary or inverse principle, according as K is of positive or negative type, but not necessarily bounded. Some of these results, for Ω a finite set, are to be found in G. Choquet and J. Deny, J. Analyse Math. **5** (1956/57), 77–135 [MR **19**, 405], where the various principles are enunciated.

G. A. Hunt (Princeton, N.J.)

12317:

Itô, Kiyosi; McKean, H. P., Jr. Potentials and the random walk. Illinois J. Math. **4** (1960), 119–132.

Etude approfondie d'un cas très particulier mais impo[illegible]tant des théories de Hunt [même J. **1** (1957), 44–93, 316–[illegible]; **2** (1958), 151–213; MR **19**, 951; **21** #5824] et Doob [J. Math. Mech. **8** (1959), 433–458; MR **21** #5825]. Il s'agit de la théorie du potentiel associée au mouvement

brownien sur Z^s; le noyau est l'opposé de l'inverse de l'opérateur de Green (l'analogue du Laplacien), à savoir l'opérateur de convolution G défini par $Gf(a)=Mf(a)-f(a)$, où $Mf(a)$ est la valeur moyenne de la fonction f sur l'ensemble des $2s$ points $a\pm e_k$, les e_k ($k=1, 2, \cdots, s$) étant les s vecteurs coordonnées. A propos de la théorie du potentiel par rapport à un noyau de convolution sur un groupe abélien localement compact, à signaler aussi une étude plus ancienne, sans interprétations probabilistes [J. Deny, Ann. Inst. Fourier Grenoble **3** (1951), 73–101; MR **16**, 698].

On donne d'abord quelques résultats standards tel que décomposition de Riesz pour les fonctions excessives positives, principe du maximum, propriétés de la capacité des ensembles; on donne aussi une démonstration nouvelle d'un résultat de B. H. Murdoch [*Preharmonic functions*, Thesis, Princeton, 1952] affirmant que toute fonction "harmonique" positive est constante; on établit un critère du type de Wiener pour qu'un ensemble infini $B\subset Z^s$ admette une distribution d'équilibre ou, ce qui revient au même, pour que la probabilité de rencontrer B une infinité de fois soit nulle; on applique enfin ce critère à la construction d'exemples effectifs. *J. Deny* (Paris)

12318:

Kishi, Masanori. On a topological property of a family of potentials. Japan. J. Math. **29** (1959), 26–31.

L'auteur étudie en espace localement compact non compact la théorie du potentiel avec un noyau symétrique $\Phi(P, Q)>0$, fini pour $P\neq Q$, et plus spécialement les rapports entre certaines principes (principe du maximum de Cartan, principe de détermination des masses par les potentiels) et l'approximation possible de toute fonction finie continue sur un compact K par une fonction finie continue qui sur K soit un potentiel sauf sur un ensemble de mesure nulle pour toute mesure d'énergie finie. Cas particulier du noyau partout fini. *M. Brelot* (Paris)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

12319:

Angheluță, Th. L'équation fonctionnelle de la translation. Inst. Politehn. Cluj. Lucrări Ști. **1959**, 29–31. (Romanian. Russian and French summaries)

For the functional equation $z[z(x, y), t]=z(x, y+t)$, the author obtains J. Aczél's solution $z(x, y)=F^{-1}[F(x)+y]$. The method is essentially the same as that of Aczél [Acta Univ. Szeged Sect. Sci. Math. **13** (1950), 179–189; MR **13**, 246], but the author says it is simpler.

I. M. H. Etherington (Edinburgh)

12320:

Angheluță, Th. Remarques concernant l'équation fonctionnelle de Poisson. Inst. Politehn. Cluj. Lucrări Ști. **1959**, 33–39. (Romanian. Russian and French summaries)

It is shown that the solutions other than $f(x)=0$ of Poisson's functional equation

$$(1)\qquad f(x+y)+f(x-y)=2f(x)f(y)$$

have the algebraic addition theorem

$$(2)\qquad \{f(x+y)\}^2-2f(x)f(y)f(x+y)+\{f(x)\}^2+\{f(y)\}^2-1=0.$$

Conversely, the solutions of (2) other than $f(x)=-\frac{1}{2}$ satisfy (1); these include the common trivial solution $f(x)=1$. It is known [P. Montel, Ann. Sci. École Norm. Sup. (3) **48** (1931), 65–94] that a continuous function of a real variable having an algebraic addition theorem is analytic, so this gives an alternative proof of Montel's independently proved result [loc. cit.] that the continuous real solutions of (1) are analytic functions. The author then solves (2) after replacing it by a differential equation and verifies that the solutions other than constants are the known solutions $\cos ax$, $\cosh ax$ of (1). There are further observations relating to É. Picard's more elementary method for such problems [*Leçons sur quelques équations fonctionelles*, Gauthier-Villars, Paris, 1928; pp. 1–2, 9–11] and to Cauchy's equation $f(x+y)=f(x)+f(y)$.

I. M. H. Etherington (Edinburgh)

12321:

Angheluță, Th. Sur une équation fonctionnelle. Inst. Politehn. Cluj. Lucrări Ști. **1959**, 17–20. (Romanian. Russian and French summaries)

The equation discussed is

$$(1)\qquad f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)=2h^3g(x).$$

This is the case $p=2$ of

$$(2)\qquad \sum_{j=0}^{2p}(-1)^j(C_{2p-1}^j-C_{2p-1}^{j-1})f\{x+(p-j)h\}=2h^{2p-1}g(x),$$

where C_n^r is the coefficient of x^r in the expansion of $(1+x)^n$. A similar equation is

$$(3)\qquad \sum_{j=0}^{2p}(-1)^jC_{2p}^jf\{x+(p-j)h\}=h^{2p}g(x).$$

The functions $f(x)$ and $g(x)$ being assumed continuous, it is proved concerning (1) and asserted concerning (2) and (3) that the equations are satisfied if and only if $f(x)$ is a polynomial of degree at most 4, $2p$, $2p+1$, respectively. {For C_2^{j-1} in line 3 read C_3^{j-1}.}

I. M. H. Etherington (Edinburgh)

12322:

Rosenbaum, R. A.; Segal, S. L. A functional equation characterising the sine. Math. Gaz. **44** (1960), 97–105.

The authors consider the functional equation

$$(1)\qquad f(z+w)f(z-w)=f(z)^2-f(w)^2$$

and prove first that the most general complex-valued functions defined over all complex numbers, satisfying (1) and being continuous at a point, bounded on every closed set and such that the set of non-zeros of f is either empty or bounded away from zero and such that

$$\lim_{z\to 0}\lim_{n\to\infty}(nz^{-1})f(zn^{-1})$$

exists, are $f(z)=az$ and $f(z)=a\sin bz$, a and b being arbitrary complex constants. Second, it is proved that the most general real-valued continuous bounded function defined over all real numbers and satisfying (1), is $f(z)=az$, $f(z)=a\sin bz$ and $f(z)=a\sinh bz$, a and b being arbitrary real constants.

As a matter of fact, (1) has been investigated and completely solved by R. D. Carmichael [Amer. Math. Monthly **16** (1909), 180–183] and W. H. Wilson [Bull. Amer. Math. Soc. **26** (1919–20), 300–312; cf. also L. Vietoris, J. Reine Angew. Math. **186** (1944), 1–15; MR **6**, 271], who have solved also the equation $g(x+y)g(x-y)=g(x)^2+g(y)^2-1$, which is posed by the authors as a problem. In their proof, the authors need the solution of the additive functional equation (2) $F(z+w)=F(z)+F(w)$ for complex-valued functions of complex variables and ask whether this equation has been already considered for complex variables. In fact, N. H. Abel [J. Reine Angew. Math. **1** (1826), 311–339; Œuvres complètes I, Christiania, 1881, pp. 219–250] has solved the equation

$$G(z+w) = G(z)G(w)$$

for complex-valued functions of complex variables, from which the solution of the simpler equation (2) follows.

J. Aczél (Debrecen)

12323:

Schweizer, Berthold. Remarks on a functional equation. J. Indian Math. Soc. (N.S.) **23** (1959), 97–99 (1961).

Basing his work on results of M. A. McKiernan [C. R. Acad. Sci. Paris **246** (1958), 2331–2334, 2564–2567; MR **20** #1784, 1785], the author finds integral solutions $f(z)$ of the functional equation $f(az)=f(z)g(z)$ in a slightly simpler way than it had formerly been done by V. Ganapathy Iyer [J. Indian Math. Soc. (N.S.) **20** (1956), 283–290; MR **19**, 642]. This method, however, does not guarantee that all the integral solutions have been found. In fact, in the formula for $f(z)$, the multiplicative constant is omitted.

M. Kuczma (Kraków)

SEQUENCES, SERIES, SUMMABILITY

See also 12111, 12193, 12208, 12345, B12554.

12324:

Verblunsky, S. A uniqueness theorem for the exponential series of Herglotz. Proc. Cambridge Philos. Soc. **56** (1960), 220–232.

Let $D(z)=1-\int_0^1 k(u)e^{zu}du$, where $k(u)$ is of bounded variation and possibly complex, $k(0)=k(0+)$, $k(1)=k(1-)$, $k(1)\neq 0$. Let $0\leq \arg k(1)<2\pi$. Then the zeros of $D(z)$ with $y\geq 0$ can be written as $z_n=x_n+iy_n$, $n=1, 2, \cdots$, where

$$x_n = \log 2n\pi - \log|k(1)| + \varepsilon_n,$$
$$y_n = 2(n+s)\pi - \arg k(1) + \tfrac{1}{2}\pi + \varepsilon_n',$$

where s is a fixed integer and ε_n, $\varepsilon_n'\to 0$ as $n\to\infty$. Further, the zeros of $D(z)$ with $y<0$ can be written as $\zeta_n=\xi_n+i\eta_n$, $n=1, 2, \cdots$, where

$$\xi_n = \log 2n\pi - \log|k(1)| + \varepsilon_n'',$$
$$\eta_n = -2(n+t)\pi - \arg k(1) - \tfrac{1}{2}\pi + \varepsilon_n''',$$

where t is a fixed integer and ε_n'', $\varepsilon_n'''\to 0$ as $n\to\infty$.

We introduce the following condition: (A) There is an $\varepsilon>0$ and an interval whose right end is 1 in which the total variation of $k(u)-k(u-v)$ is $O(v^\varepsilon)$ as $v\to+0$.

Uniqueness theorem: Let $k(u)$ satisfy condition (A). Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of numbers such that $\sum \alpha_n \exp(-z_n t)$, $\sum \beta_n \exp(-\zeta_n t)$ converge in (0, 1), and let

$$\sum \alpha_n \exp(-z_n t) + \sum \beta_n \exp(-\zeta_n t) = f(t),$$

where $f\in L(0, 1)$. Then for $n=1, 2, \cdots$

$$\alpha_n = \int_0^1 f(v)w_n^{(1)}(v)\,dv, \quad \beta_n = \int_0^1 f(v)w_n^{(2)}(v)\,dv,$$

where $w_n^{(1)}(v)$, $w_n^{(2)}(v)$ are obtained from

$$w_n(v) = w_n(v;t) = -e^{\mu_n t}\operatorname*{res}_{\mu_n} \frac{e^{z(v-t)}}{D(z)} \int_{1-v}^1 k(u)e^{zu}\,du$$

on replacing μ_n by z_n, ζ_n respectively.

E. Følner (Copenhagen)

12325:

Yano, Kenji. A note on absolute Cesàro summability of Fourier series. Tôhoku Math. J. (2) **12** (1960), 293–300.

Let $p\geq 1$, $f\in L^p$, then for $t>0$

$$w_p(t) = \left((2\pi)^{-1}\int_0^{2\pi} |f(x+t)-f(x)|^p\,dx\right)^{1/p}.$$

The author proves the following. If $1<p\leq 2$, $f\in L^p$ and $\int_0^\pi w_p^*(t)/t\,dt<\infty$ for a function $w_p^*(t)\geq w_p(t)$, then the Fourier series of f is summable $|C, 1/p|$ almost everywhere, provided that $\lambda(x)=[w_p^*(1/x)\log x]^{-1}$ is a function with the properties: For $0<x_0\leq x$, $\lambda(x)>0$ with $\lambda(x)\uparrow\infty$ as $x\uparrow\infty$ and for a positive constant H depending only on δ, $H<\lambda(x^\delta)/\lambda(x)\leq 1$ for $0<\delta<1$ and $x\geq x_0$. A second similar theorem and two corollaries are stated, and some remarks concerning relations to results of H. C. Chow [J. London Math. Soc. **30** (1955), 439–448; MR **17**, 32] and F. T. Wang [Duke Math. J. **9** (1942), 567–572; MR **4**, 37] are made.

G. Goes (Evanston, Ill.)

12326:

Boyd, A. V. Multiplication of strongly summable series. Proc. Glasgow Math. Assoc. **4**, 29–33 (1958).

Denoting the nth Cesàro sum and mean of order k ($k>-1$) of an infinite series $\sum a_n$ by $A_n^{(k)}$ and $a_n^{(k)}$ respectively, we have the following definitions: summability (C, k) to $s: a_n^{(k)}\to s$; summability $(C, -1)$ to $s: a_n^{(0)}=A_n^{(0)}\to s$ and $na_n=o(1)$; absolute summability (C, k), or summability $|C, k|$ to $s: a_n^{(k)}\to s$ and

$$\sum_{n=1}^{\infty} |a_n^{(k)}-a_{n-1}^{(k)}| < \infty;$$

strong summability with order $k>0$ and index p, or summability $[C; k, p]$ to $s: \sum_{n=0}^N |a_n^{(k-1)}-s|^p=o(N)$. Defining summability $[C; 0, p]$ as convergence together with the condition: $\sum_{n=0}^N n^p|a_n|^p=o(N)$, Hyslop [same Proc. **1** (1952), 16–20; MR **14**, 368] proved the inclusion: for $k\geq 0$, $p\geq 1$, $[C; k, p]\subset[C; k+\delta, q]$ for every $\delta>0$ and $q\leq p$. It is also known that for $k\geq 0$, $|C, k|\subset[C; k, 1]$, but the converse is not necessarily true. Also $(C, k-1)\subset[C, k]$ for $k\geq 0$ [Winn, Math. Z. **37** (1933), 481–492]. For brevity we use hereafter the following symbolism: by $A\times B\Rightarrow C$ we mean that if any series $\sum u_n$ is summable A and any other series $\sum v_n$ is summable B, then the series $\sum w_n=\sum(u_0v_n+u_1v_{n-1}+\cdots+u_nv_0)$ is summable C, and further, the product of the sums is the sum of the "product series". We have the following results from Winn (ibid.): (i) for $k>0$, $l\geq 0$, $[C, k]\times(C, l)\Rightarrow(C, k+l)$; (ii) for $k>0$, $l>0$, $[C, k]\times[C, l]\Rightarrow[C, k+l]$.

With regard to these results of Winn, the author proves:

(I) Theorem (ii) of Winn breaks down in the case $k=l=0$ in the sense that $[C, 0]\times[C, 0]\not\subset[C, 0]$. The counter-example used to demonstrate this is: $u_0=u_1=0$; $u_n=(-1)^n/n \log n$ $(n\geq 2)$; $v_0=v_1=v_2=0$; $v_n=(-1)^n/(n \log\log n)$ $(n\geq 3)$. Both $\sum u_n$ and $\sum v_n$ are summable $(C, -1)$ and hence summable $[C, 0]$, but $\sum w_n$ is not summable $[C, 0]$.

(II) $[C, 0]\times[C, 0]\Rightarrow(C, 0)$.

(III) $[C, 0]\times(C, 0)\not\subset[C, 1]$. The counter-example used to prove this has the same u_n, but a more complicated v_n. The author first shows that the $\sum w_n$ is not summable $[C, 1]$ to st, the product of the sums; and in view of its summability $[C, 1+\delta]$ to st (an easy deduction from theorem (ii) of Winn) and the necessity of the $[C, 1]$-sum being st, infers that $\sum w_n$ is not summable $[C, 1]$. Since $(C, 0)\subset[C, 1]$, the following are immediate from (III): (IIIa) $[C, 0]\times(C, 0)\not\subset(C, 0)$; (IIIb) $[C, 0]\times[C, 1]\not\subset[C, 1]$. Thus Winn's (i) and (ii) cannot be extended to the cases: $k=l=0$; $k=0$, $l=1$.

(IV) Further, to quote the interesting reasoning of the author, if $\sum v_n$ were summable $[C, 1-\delta]$ for some $\delta>0$, then, since $\sum u_n$ is summable $[C, \frac{1}{2}\delta]$, $\sum w_n$ would be summable $[C, 1-\frac{1}{2}\delta]$ and hence summable $[C, 1]$. Since this is not the case, it follows that $\sum v_n$ is not summable $[C, 1-\delta]$, although it is summable $(C, 0)$ and therefore $[C, 1]$. This shows that, when $k=1$, Winn's result: $(C, k-1)\subset[C, k]$ $(k>0)$ (which also holds for $k=0$) is in a sense "best possible".

Finally, the author gives the 'multiplication theorem': (V) for $k>0$, $[C, k]\times|C, 0|\Rightarrow[C, k]$, and leaves the question whether the result is still true for $k=0$ open. It may be recalled that summability $|C, 0|$ is, by definition, the same as absolute convergence. *T. Pati* (Allahabad)

12327:

Posner, Edward C. Summability-preserving functions. Proc. Amer. Math. Soc. **12** (1961), 73–76.

Gegeben seien die Toeplitz-Matrix $A=(a_{ik})$ und eine in der komplexen Ebene erklärte Funktion f. Für jede beschränkte C_1-limitierbare Folge $\{x_n\}$ sei $\{f(x_n)\}$ A-limitierbar. Dann ist f notwendig ganzlinear. Beweis: (a) Zunächst zeigt man, daß A die Folge 1, 0, 1, 0, $\cdots$ limitiert, so daß

$$\lim_{i\to\infty}\sum_{k \text{ ungerade}} a_{ik}=r \text{ und } \lim_{i\to\infty}\sum_{k \text{ gerade}} a_{ik}=1-r$$

existieren. (b) Durch Hintereinanderlegen von hinreichend langen Blöcken der Form $a, b, a, b, \cdots, a, b$ und $\frac{1}{2}(a+b), \frac{1}{2}(a+b), \cdots, \frac{1}{2}(a+b)$ (a, b, beliebig) wird gezeigt, daß f der Funktionalgleichung $rf(a)+(1-r)f(b)=f(\frac{1}{2}(a+b))$ genügt. Vertauscht man a, b, so kommt $r=\frac{1}{2}$, und mit der Stetigkeit von f die Behauptung. *D. Gaier* (Giessen)

12328:

Jha, D. K. On translative matrices for sequence-to-sequence transformations. Bull. Soc. Math. Belg. **12** (1960), 57–72.

Sufficient conditions are obtained that a T-matrix (regular matrix) should be translative for bounded sequences. There is a mistake in Lemma 4, p. 60 (observed by the author after publication of the paper); the words "and only if" should be deleted from line 3 of the enunciation and from p. 61, line 9. For $\{\rho_n\}$ is not an arbitrary convergent sequence, but is the transform of $\{\delta_n\}$ by the matrix A. In consequence the words "necessary and" should be deleted from the enunciation of Theorem 1, together with lines 4–8 from the bottom of p. 61. Next, when a K-matrix (conservative matrix) is efficient for all bounded sequences, a necessary and sufficient condition is obtained in order that the matrix should be translative for such sequences. Two classes of row-finite matrices $A(\alpha_p)$ and $L(\beta_p)$ (the members of the first being upper-semi, and those of the second lower-semi, matrices) which are translative, are also considered. It is proved that all T-matrices translative to the right [left] are consistent for sequences in the summability field of any one or more of the matrices of the class $A(\alpha_p)$ [$L(\beta_p)$], respectively. It is next shown that all T-matrices which are translative for all periodic sequences are consistent for such sequences, but that the converse is false. It is finally proved that if a translative K-matrix sums at least one periodic sequence to its arithmetic mean (assumed not zero), then it is a T-matrix. An example is given to show that there are T-matrices which are absolutely translative for all bounded sequences, but are not absolutely equivalent for all such sequences. *R. G. Cooke* (London)

12329:

Endl, K. Untersuchungen über Momentenprobleme bei Verfahren vom Hausdorffschen Typus. Math. Ann. **139**, 403–432 (1960).

The author considers summability methods corresponding to matrices of the form $H^{(\alpha)}(\mu)=\delta^{(\alpha)}\mu^{\delta^{(\alpha)}}$ where $\mu=(\mu_n)$ is a diagonal matrix and $\delta^{(\alpha)}=\left((-1)^n\binom{m+\alpha}{m-n}\right)$, α real. The case $\alpha=0$ yields the class of Hausdorff methods. The Toeplitz regularity conditions for $H^{(\alpha)}(\mu)$ become

$$\sup_m \sum_{n=0}^{m}\left|\binom{m+\alpha}{m-n}\Delta^{m-n}\mu_n\right|<\infty,$$

$$\lim_{m\to\infty}\sum_{n=0}^{m}\binom{m+\alpha}{m-n}\Delta^{m-n}\mu_n=1,$$

$$\lim_{m\to\infty}\binom{m+\alpha}{m-n}\Delta^{m-n}\mu_n=0.$$

Typical results of this comprehensive and detailed treatment of the problem include the following: (i) the method $H^{(\alpha)}(\mu)$ is regular if and only if there is a function $\chi(x)$ of bounded variation with $\int_{0+}^{1}x^{-\alpha}|d\chi|<\infty$, $\int_{0+}^{1}x^{-\alpha}d\chi=1$, such that $\mu_n=\int_0^1 x^n d\chi$; (ii) for fixed α the methods $H^{(\alpha)}(\mu^{(1)})$, and $H^{(\alpha)}(\mu^{(2)})$, corresponding to any two diagonal matrices $\mu^{(1)}$ and $\mu^{(2)}$, are permutable; (iii) for a given μ, one of the following holds: $H^{(\alpha)}(\mu)$ is not convergence-preserving for any α; there exists $\alpha^*=\alpha^*(\mu)$ such that is convergence-preserving for $\alpha<\alpha^*$ and is not convergence-preserving for $\alpha>\alpha^*$.

W. T. Scott (Evanston, Ill.)

12330:

Petersen, G. M. Matrices and norms. Proc. Cambridge Philos. Soc. **57** (1961), 271–273.

The norm $L(A)$ of a regular T-matrix $A=(a_{m,n})$ is defined by $L(A)=\limsup_m \sum_{n=1}^{\infty}|a_{m,n}|$. If A sums all bounded sequences that are summable by $B=(b_{m,n})$, A is said to be b-stronger than B. The norm of a matrix $\mathfrak{A}$ is

defined by $\|\mathfrak{A}\| = \inf h(A)$, where the inf is taken over all the matrices equivalent to $\mathfrak{A}$ for bounded sequences. These norms have been studied by A. Brudno [Mat. Sb. (N.S.) **16** (58) (1945), 191–247; MR **7**, 12], one of whose main results is that if $\mathfrak{B}$ sums all bounded sequences that are $\mathfrak{A}$-summable, then $\|\mathfrak{B}\| \geq \|\mathfrak{A}\|$. In the present paper, the author proves the following theorem. Let A be regular and be absolutely translative for all bounded sequences, i.e., $\lim_{m\to\infty} \sum_{n=1}^{\infty} |a_{m,n} - a_{m,n+1}| = 0$. Then if $\|\mathfrak{A}\| > 1$, for each θ of an everywhere dense set in the interval $(1, \|\mathfrak{A}\|)$, there is a matrix B such that A is b-stronger than B, and $\|\mathfrak{B}\| = \theta$. *R. G. Cooke* (London)

12331:

Coomes, H. R.; Cowling, V. F. Summability and associative infinite matrices. Michigan Math. J. **8** (1961), 65–70.

Sequence-to-sequence matrix transformations $y_n = \sum_{k=0}^{\infty} a_{n,k} x_k$ $(n = 0, 1, 2, \cdots)$ are considered; $A = (a_{n,k})$ is conservative (or a K-matrix) if y_n tends to a limit as $n \to \infty$ whenever x_k tends to a limit as $k \to \infty$. If $A, B, \cdots$ are K-matrices, the column limits are denoted by $a_k, b_k, \cdots$. A K-matrix A is said to be co-regular if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} - \sum_{k=0}^{\infty} a_k \neq 0$; otherwise it is said to be co-null. The sequences $e_k = 1$ for all k, $e_k^n = \delta_{n,k}$ $(n, k = 0, 1, 2, \cdots)$ are denoted by e, e^n; let $\Delta = \{e^n : n = 0, 1, 2, \cdots\}$, and let Φ be the set consisting of elements of Δ together with e. The authors define a matrix A to be associative if $B(Ax) = BA(x)$ for all matrices B with finite K_r-bound $\|B\|$ and all x in the summability field C_A. The matrix A is said to be replaceable if there exists a regular matrix D such that $C_D = C_A$. A K-matrix A is said to have maximal inset if $\sum a_k x_k$ converges for all x in C_A, and is said to have propagation of maximal inset (PMI) if $\sum b_k x_k$ converges for all x in C_A whenever B is a matrix such that $C_B = C_A$ [see A. Wilansky, Duke Math. J. **19** (1952), 647–660; MR **14**, 369; M. S. MacPhail, Canad. J. Math. **6** (1954), 405–409; MR **15**, 949]. Following on a number of lemmas, the theorems given below are proved. Theorem 1: Let A be a replaceable matrix, then the following are equivalent: (1) Φ is a basis for C_A; (2) A is associative; (3) A has PMI. Theorem 2: Let A be a co-regular matrix, then Φ is a basis for C_A if and only if A has PMI. Theorem 3: Let A be a K-matrix, then Δ is a basis for C_A if and only if (1) A is associative and (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} x_k = \sum_{k=0}^{\infty} a_k x_k$ for all x in C_A. *R. G. Cooke* (London)

12332:

Reĭmers, Ė. G. Mean value theorems for double series. Tartu Riikl. Ül. Toimetised **62** (1958), 60–79. (Russian. Estonian and English summaries)

Der Verfasser verallgemeinert Untersuchungen von Jurkat und dem Referent [Jurkat und Peyerimhoff, Math. Z. **55** (1951), 92–108; **56** (1952), 152–178; MR **13**, 934; **14**, 158, **15**, 1139; Peyerimhoff, ibid. **57** (1953), 265–290; MR **14**, 865] auf Doppelfolgen. Für verschiedene Konvergenzdefinitionen (auch absolute Konvergenz) bei Doppelfolgen wird Summierbarkeit eingeführt, in den Wirkfeldern (die als Banachräume aufgefasst werden können) werden Begriffe Abschnittskonvergenz, schwache Abschnittskonvergenz und Perfektheit erklärt. Es werden ferner drei Mittelwertsätze definiert und ihr Zusammenhang mit diesen Begriffen untersucht. Für diese Mittelwertsätze werden hinreichende Bedingungen angegeben (Diskussion der Verfahren von Riesz, Cesàro und Nörlund-Voronoĭ). Schliesslich werden diese Mittelwertsätze verwendet, um Vergleichssätze zu beweisen. *A. Peyerimhoff* (Marburg)

12333:

Reĭmers, Ė. Mean value theorems and multiplication of double summable series. Tartu Riikl. Ül. Toimetised **73** (1959), 50–83. (Russian. Estonian and English summaries)

Die Arbeit schliesst sich an die im vorangehenden Referat [#12332] besprochene Arbeit an. Für Doppelreihen $\sum u_{kl}$, $\sum v_{kl}$, $\sum w_{kl}$, $w_{kl} = \sum_{\mu=0}^{kl} v_{k-\mu l-\nu} u_{\mu\nu}$ wird die Frage nach der C-Summierbarkeit von $\sum w_{kl}$ als Folge der A- bzw. B-Summierbarkeit von $\sum u_{kl}$ bzw. $\sum v_{kl}$ untersucht (A und B gegeben, wie muss C beschaffen sein und A = absolute Konvergenz und C gegeben, wie muss B beschaffen sein). Es ist nicht möglich, im Rahmen eines Referates auf Einzelheiten einzugehen (für den Fall gewöhnlicher Reihen vgl. Reĭmers, Dokl. Akad. Nauk SSSR **120** (1958), 1196–1199 [MR **20** #4720]). Es werden Anwendungen auf die Verfahren von Riesz und Nörlund-Voronoĭ besprochen. *A. Peyerimhoff* (Marburg)

12334:

Subhankulov, M. A. Tauberian theorems with remainder term. Mat. Sb. (N.S.) **52** (**94**) (1960), 823–846. (Russian)

The author gives several Tauberian theorems for power series, for Dirichlet series, and for integrals, some of which generalize the results of A. G. Postnikov [Dokl. Akad. Nauk SSSR **77** (1951), 193–196; **92** (1953), 487–490; MR **12**, 820; **15**, 951]. A prototype of the results given is the following classical theorem. If b_m is A summable to 0, i.e., if $f(z) = \sum_{m=0}^{\infty} b_m z^m \to 0$ for $z \to 1-$ and if $b_m \geq 0$, then b_m is C_1 summable to 0: $B_k = \sum_0^k b_m = o(k)$. Two examples are quoted: (1) If $b_m > -Cm^{\gamma-1/\beta}\psi_1(m)\psi_2(m)$, if $f(z)$ exists for $|z| < 1$ and satisfies

$$f(z) = O(|1-z|^{-\alpha}\psi_3((1-r)^{-1}))$$

in a region with the boundary around $z = 1$ given by $|\theta| = (1-r)^{1/\beta}\psi_2((1-r)^{-1})$, then

$$B_k = O(k^{\gamma+1/\beta}\psi_1(k)/\psi_2(k) + k^{\alpha}\psi_3(k)).$$

Here $z = re^{i\theta}$, $\beta \geq 1$, $0 \leq \alpha \leq \gamma - (1/\beta) + 1 < 3$, $1/\beta < \gamma$ and the $\psi_i(k)$ are certain functions increasing slower than k^s for each $s > 0$. (2) If $|b_m| < Cm^{\gamma-1/2}\psi_1(m)$, $\frac{1}{2} < \gamma \neq 1$ and $g(z) = \sum_{m=0}^{\infty} a_m z^m$, $|a_m| < C_1 m^{\alpha-1/2}\psi_2(m)$, $\alpha > \frac{1}{2}$ satisfy $f(z) = O(|1-z|^{\gamma}\psi_1((1-r)^{-1}))$, $g(z) = O((1-r)^{-\alpha}\psi_2((1-r)^{-1}))$ for $0 < r < 1$, $|\theta| < \text{const}$, then $\sum_0^k a_m b_m = O(k^{\alpha+\gamma-1}\psi_3(k))$. *G. G. Lorentz* (Syracuse, N.Y.)

12335:

Takagi, Shigeyuki. A Tauberian theorem for Fourier series. Proc. Japan Acad. **35** (1959), 417–420.

Let $\phi(t)$ be an even Lebesgue integrable periodic function of period 2π and let $\frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nt$ be its Fourier series. It is proved that if $\sum a_n$ is summable in the Abel sense to the value 0 and $\sum_{\nu=n}^{2n} |a_\nu| = o(1/\log n)$ and if, for some positive constants t_0, ρ and A, $\phi'(t) > -At^{-\rho}$ for $0 < t < t_0$, then $\phi(t) \to 0$ as $t \to 0$.

This may be regarded as an analogue of a theorem of Hardy and Littlewood [Ann. Scuola Norm. Sup. Pisa (2)

3 (1934), 43–62] in which the roles of $\phi(t)$ and $s_n = \frac{1}{2}a_0 + \sum_{\nu=1}^{n} a_\nu$ are reversed. *H. R. Pitt* (Nottingham)

12336:
Rajagopal, C. T. Remarks on a theorem of Kuttner's. Quart. J. Math. Oxford Ser. (2) **11** (1960), 258–262.

The paper deals with the Mercer transformation

$$\theta(x) = \lambda f(x) + (1-\lambda)g(x), \quad g(x) = x^{-1}\int_0^x f(t)\,dt$$

of a function $f(x)$ bounded and integrable in every finite interval of $x \geq 0$ in the exceptional case in which the real part of λ, but not λ itself, vanishes. The kernel $N(x)$ is non-negative and bounded in every finite interval in $x>0$, $N(x)\log x$ is integrable in $(1, \infty)$, and $K(x) = \int_x^\infty N(t)dt$, $K(0)=1$. Then it is shown that $\Psi(s) = s\int_0^\infty N(st)f(t)dt$ exists for $s>0$ and

$$\limsup_{s\to+0} |\alpha^{-1}\Psi(s) - (1-\alpha^{-1})K_\alpha(0)g(s^{-1})| \leqq \tau_\alpha \limsup_{x\to\infty} |\theta(x)|,$$

where

$$\alpha = \lambda^{-1}, \quad K_\alpha(x) = \int_x^\infty t^{-\alpha}N(t)\,dt,$$

$$\tau_\alpha = \int_0^1 |N(t) + (\alpha-1)t^{\alpha-1}\{K_\alpha(0) - K_\alpha(t)\}|\,dt + \int_1^\infty |N(t) - (\alpha-1)t^{\alpha-1}K_\alpha(t)|\,dt.$$

Moreover, the constant τ_α is best possible and equality may be attained.

The limiting case obtained by putting $\alpha = 0$ is also valid and is a special case of a theorem already proved by the author [Comment. Math. Helv. **30** (1954), 63–72; MR **17**, 255]. In the case $\theta(x)\to 0$, it follows, under the extra condition $K_\alpha(0)\neq 0$, that the convergence of $\Psi(s)$ implies that of $g(x)$ and $f(x)$.

The case $N(x) = e^{-x}$ was treated by Kuttner [Quart. J. Math. Oxford Ser. (2) **11** (1960), 151–160; MR **22** #6963], and the limiting value $\alpha = 0$ then gives Tauber's second theorem [Hardy, *Divergent series*, Clarendon, Oxford, 1949; MR **11**, 25; p. 150, Theorem 86].

By combining these results with a theorem of the Wiener type, the author deduces that if $\int_0^\infty t^{iu}N(t)dt \neq 0$ for real u, the convergence of $\Psi(s)$ implies the summability (C, ε) of $f(x)$ for every positive ε.

H. R. Pitt (Nottingham)

APPROXIMATIONS AND EXPANSIONS

12337:
Murnaghan, Francis D. The approximation of differentiable functions by polynomials. An. Acad. Brasil. Ci. **31** (1959), 25–29.

Some remarks on the numerical determination of the Chebyshev polynomial (the polynomial of degree n, $p_n(x)$, which minimizes $\max_{a\leq x\leq b} |f(x) - p_n(x)|$) for a differentiable function $f(x)$. The author refers to an iterative procedure developed by him and Wrench [David Taylor Model Basin, R and D Report 1175 (1958)]. Numerical values are given for arc tan x and cos $\pi x/4$, $-1 \leqq x \leqq 1$, with $n=6$ and 3, respectively.

P. J. Davis (Washington, D.C.)

12338a:
Walsh, J. L. Note on polynomial approximation on a Jordan arc. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 981–983.

12338b:
Walsh, J. L. Degree of approximation by bounded harmonic functions. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1390–1393.

In the first paper, conformal mapping is applied to approximation problems. Recently the author proved the following. Let the positive sequence (ε_n) satisfy certain conditions and tend to zero as $n\to\infty$, $r^n = O(\varepsilon_n)$ for any fixed r $(0<r<1)$, $f(z)$ be defined on the Jordan arc E, while $f_n(z)$ $(n=1, 2, \cdots)$ is analytic in the region $D \supset E$. If

(1) $|f(z) - f_n(z)| \leqq A_1\varepsilon_n$ on E, (2) $|f_n(z)| \leqq A_2R^n$ in D

for some $R>0$, then there are polynomials $p_n(z)$ of degrees n such that

(3) $$|f(z) - p_n(z)| \leqq A_3\varepsilon_n \text{ on } E.$$

The A's are independent of n and z. Now it is shown that, for bounded D, (1) and (2) are necessary conditions.

Since the hypothesis is invariant under a one-one conformal mapping $w = w(z)$ of E to E_1 and, consequently, of some sub-region of D to $D_1 \supset E_1$, the classes of functions f that can be approximated by polynomials on E or E_1, respectively, with degree of approximation $O(\varepsilon_n)$, are identical. Hence E_1 may be taken as the interval E_0: $-1 \leqq x \leqq 1$, and so by known results on trigonometric approximations $f(z)$ is approximated on E by polynomials in z, with degree of approximation $O(n^{-k-\alpha})$ $(0<\alpha<1)$, if and only if $d^kf[\phi(\cos\theta)]/d\theta^k$ satisfies a Lipschitz condition of order α, where $z=\phi(x)$ maps E_0 on E one-to-one conformally.

Finally (Theorem 3) a result is deduced concerning polynomial approximation, with degree $O(\varepsilon_n)$, on an analytic Jordan arc E.

In the second paper, the results which the author proved in the first paper are extended to more general degrees of approximation and to further cases of Jordan arcs.

Theorem 1: Let E be an analytic Jordan curve containing $z=0$ in its interior, $u(z)$ be defined on E, D and the $\{\varepsilon_n\}$ and the A's as in the first paper, the $u_n(z)$ harmonic in D. If

(1) $|u(z) - u_n(z)| \leqq A_1\varepsilon_n$ on E, (2) $|u_n(z)| \leqq A_2R^n$ in D,

then there are harmonic polynomials $p_n(z, 1/z)$ in z and $1/z$ of degrees n such that

(3) $$|u(z) - p_n(z, 1/z)| \leqq A_3\varepsilon_n \text{ on } E.$$

A corollary and its converse, Theorem 2, deal with the case when D contains both E and its interior. A further corollary gives a simplification of Theorem 1. Then an analytic Jordan arc E is considered. Let u_n be defined as above, $D \supset E$, and (1) and (2) be satisfied, let $z=\phi(w)$ map E one-to-one and conformally onto E_0: $-1 \leqq w \leqq 1$. Then the function $u[\phi(\cos\theta)]$ can be approximated by trigonometric polynomials in θ of degrees n, with degree of approximation $O(\varepsilon_n)$.

In Theorem 4 (cf. Theorem 3 of the first paper)

necessary and sufficient conditions, in terms of approximation by trigonometric polynomials, are given for the existence of the $u_n(z)$ satisfying (1) and (2).

H. Kober (Birmingham)

12339:

Remez, E. Ya. Sur les points d'écart fixes des solutions des problèmes d'approximation de Tchebycheff aux paramètres entrant linéairement. II. Ukrain. Nat. Ž. **9** (1957), 310–328. (Russian. French summary)

(Continuation of the review of part I [same Ž. **9** (1957), 44–65; MR **19**, 544].) Another, and characteristic property of the lowest face, given by the transposition theorem for linear inequalities, becomes in the present setting: z is in Z if and only if a positive combination of finitely many (say, h) terms of the form $(\sum a_k z_k)\operatorname{sgn}\overline{\sum a_k z_k - a_0}$, a in $A(z)$, vanishes [Remez, ibid. **5** (1953), 3–49; MR **15**, 407], and the set A_1 of all a appearing in such combinations does not depend on z. For real finite A, $A_1 = A_0$. For $h > 2n+1$ ($h > n+1$ for real A) some term can be dropped. The author applies the real case to problems of W. Markoff [*On the functions deviating least from zero in a given interval*, St. Petersburg, 1892; Jahrb. Fortschr. Math. **24** (1892), 355; also Math. Ann. **77** (1916), 213–258] and Shohat [Ann. Ural. (Ekaterinburg) Univ. **1** (1922), 1–33]; the complex case relates to results of de la Vallée-Poussin (1911) and Fekete [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 95–103; MR **13**, 32].

T. S. Motzkin (Los Angeles, Calif.)

12340:

van der Corput, J. G. Neutrix calculus. I. Neutrices and distributions. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 115–123.

In order to solve a certain problem the author introduces a new auxiliary variable ξ, which does not occur in the given problem. Under a special arrangement the final result is found when everywhere each term which depends on ξ is ignored, so that only the constant terms are taken into account. The main point is: to arrange it in such a way that the final result is completely correct. For that purpose the author introduces functions which are called negligible. These functions are subjected to many conventions, of which the main are: First convention. The class formed by the negligible functions is an additive group. Second convention (neutrix-condition). A negligible function which assumes for each element ξ of the domain N' the same value, is identically equal to zero.

Definition: A neutrix is an additive group N formed by functions which satisfy the neutrix-condition. Fourth convention. Two neutrices are called equal if and only if they have the same domain, the same negligible functions and the same variable. The ordinary limit concept is generalised as follows. If it is possible to find a number l such that $f(\xi) - l$ is negligible in the group N, this number is called the N-limit of $f(\xi)$ for $\xi \to b$ (b is a limit point which does not belong to the domain N').

Notation: $N - \lim_{\xi \to b} f(\xi) = l$. l is uniquely defined. The main theorem in the theory of the neutrix limits can be formulated as follows: If $f_h(\xi)$ ($h = 1, 2, \cdots$) are arbitrary real or complex functions with domain N' which has a limit point b not belonging to N' such that $N - \lim_{\xi \to b} f_h(\xi)$ exists for $h = 1, 2, \cdots$ with the understanding that, whenever a function $g(\xi)$, defined on N', tends to a finite limit l as ξ tends to b on N, then l is also the N-limit of $g(\xi)$ for $\xi \to b$. The consequences are: Each limit exists, each function is differentiable, integrable, and so on. By means of neutrices the author constructs new mathematical objects which he calls distributions (since the Schwartz distributions are only special cases) in the following way: For a positive integer s he considers s distinct neutrices $N_1, \cdots, N_s$ with the same range N'' and with the domains $N_1', \cdots, N_s'$. The variables $\xi_1, \cdots, \xi_s$ corresponding to these neutrices are therefore independent. Let $f(\xi_1, \cdots, \xi_s)$ be a function with range N'', defined for each element ξ_σ of N_σ' ($\sigma = 1, \cdots, s$). Consider the class d formed by all the functions of the form $f(\xi_1, \cdots, \xi_s) + \nu_1(\xi_1) + \cdots + \nu_s(\xi_s)$ where $\nu_\sigma(\xi_\sigma)$ ($\sigma = 1, \cdots, s$) denotes an arbitrary function which is negligible in N_σ. This class d is called the distribution with the neutrices $N_1, \cdots, N_s$ generated by the function $f(\xi_1, \cdots, \xi_s)$. The order r ($0 \leq r \leq s$) is the smallest positive integer such that the distribution d possesses a generating function which depends only on r of the s variables $\xi_1, \cdots, \xi_s$.

The author ends with the development of an extensive program of which the main points are: Special neutrix calculus (theory of the distributions of order zero). Show that this theory gives as special the "finite part" of an integral, introduced by Hadamard. Simplify by means of the special matrix calculus several methods and many proofs. Obtain old results under new conditions or prove new results in a simple way. General neutrix calculus. Introduce in the theory of the distributions of arbitrary order, as far as possible, the notions of addition, subtraction, multiplication, convergent series, limit, derivative, Fourier- and Laplace-transform and so on.

He remarks: It is unnecessary to say that it is impossible for one man to carry out the complete program.

S. C. van Veen (Delft)

FOURIER ANALYSIS

See also 12186, 12325, 12335, 12386, 12389, 12391, 12392, 12394, 12402, B12595, B13340.

12341:

Babenko, K. I. Approximation of periodic functions of many variables by trigonometric polynomials. Dokl. Akad. Nauk SSSR **132** (1960), 247–250 (Russian); translated as Soviet Math. Dokl. **1**, 513–516.

Let $f(x)$ be defined for $x = (x_1, x_2, \cdots, x_m)$ in E_m, an affine m-dimensional space. Let $f(x)$ have period 1 in each variable and let $\mathfrak{H}$ denote the Hilbert space of functions $f(x)$ square-integrable over the unit cube K. Let $P_j(D) = P_j(p_1, p_2, \cdots, p_m)$ be a polynomial in $p_k = i^{-1}\partial/\partial x^k$ with constant coefficients. Let F_p be the class of functions f satisfying the condition $\sum_j \int_K |P_j(D)f|^2 dx \leq 1$. Let $Q(D) = \sum_j P_j^*(D)P_j(D)$. Let $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ be the characteristic vectors of $Q(D)$ corresponding to the characteristic values $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$. The author seeks to approximate $f \in F_p$ by a set of functions $f_1, f_2, \cdots, f_n$. Let $\rho_n^2(f) = \|f\|^2 - \sum_{k=1}^n |(f, f_k)|^2$. It is shown that

$$\inf_{f_1, f_2, \cdots, f_n} \sup_{f \in F_p} \rho_n(f) = \lambda_{n+1}^{-1/2},$$

and that the extremal functions are the characteristic vectors of $Q(D)$. Results of similar type for functions of one variable were obtained by Kolmogoroff [Ann. of Math. (2) **37** (1936), 107–110].

J. G. Herriot (Stanford, Calif.)

12342:

Babenko, K. I. Approximation by trigonometric polynomials in a certain class of periodic functions of several variables. Dokl. Akad. Nauk SSSR **132** (1960), 982–985 (Russian); translated as Soviet Math. Dokl. **1**, 672–675.

The notation of this paper is the same as that of the paper [#12341] reviewed above. The author discusses the special case in which F_p is the set of periodic functions $f(x)$ subject to the condition

$$\sup_{x\in K} |\partial^{l_1+\cdots+l_m} f(x)/\partial x_1^{l_1}\cdots\partial x_m^{l_m}| \leqq 1,$$

where $l_1, l_2, \cdots, l_m$ are integers. The operator considered is $P(D)=(-1)^{l_1+l_2+\cdots+l_m}p_1^{2l_1}\cdots p_m^{2l_m}$. Also considered is the class of functions which in K satisfy the condition

$$\left|\frac{\partial^{j_{i_1}+l_{i_1}+\cdots+l_{i_s}} f(x)}{\partial x_{j_1}^{l_{i_1}}\cdots\partial x_{j_m}^{l_{i_m}}}\right| \leqq 1,$$

where $1\leqq j_1<j_2<\cdots<j_s\leqq m$, $1\leqq s\leqq m$. An approximation theorem is then obtained. It is pointed out that this result may be applied to the problem of constructing a formula for quadrature. *J. G. Herriot* (Stanford, Calif.)

12343:

Helson, Henry; Szegö, Gabor. A problem in prediction theory. Ann. Mat. Pura Appl. (4) **51** (1960), 107–138.

Sei μ ein nicht-negatives endliches Mass auf $[0, 2\pi)$, L_μ^2 der Hilbertraum der f mit $\int |f|^2 d\mu<\infty$, $d\mu(x)=w(x)dx/2\pi+d\mu_s(x)$, wo $\mu_s(x)$ der singuläre Bestandteil ist. Mit $\mathfrak{F}$, $\mathfrak{P}$, $\mathfrak{P}_1$ werden die abgeschlossenen linearen Hüllen in L_μ^2 der trigonometrischen Polynome $F(e^{ix})=a_1e^{ix}+a_2e^{2ix}+\cdots$, $P(e^{ix})=b_1e^{-ix}+b_2e^{-2ix}+\cdots$, $P_1(e^{ix})=c_0+c_1e^{-ix}c_2e^{-2ix}+\cdots$ bezeichnet. Für zwei Mannigfaltigkeiten A und B in L_μ^2 wird $\rho=\sup|(f, g)|$, $f\in A$, $g\in B$ gesetzt; A und B heissen "at positive angle", wenn $\rho<1$ ist. Die Verfasser geben ρ an für $\mathfrak{F}$ und $\mathfrak{P}_1$, wenn $w(x)^{-1}$ bzw. $w(x)$ ein positives trigonometrisches Polynom, oder wenn $w(x)$ der Quotient zweier positiver trigonometrischer Polynome ist. Genau dann ist $\rho<1$ für $\mathfrak{F}$ und $\mathfrak{P}_1$, wenn μ absolut stetig ist und $w=e^{u+\tilde{v}}$ ist mit u beschränkt reell, $\tilde{v}$ konjugiert zu v beschränkt reell, $\|v\|_\infty<\frac{1}{2}\pi$. Ist f ein reelles trigonometrisches Polynom, so ist genau dann $\|\tilde{f}\|\leqq K\|f\|$ (K universell, $\tilde{f}$ konjugiert zu f) mit der Norm in L_μ^2, wenn $\rho<1$ ist für $\mathfrak{F}$ und $\mathfrak{P}_1$. Es folgt eine Diskussion von $\mathfrak{F}$ und $\mathfrak{P}$; u.a. führt $w_\alpha(x)=|1+e^{ix}|^\alpha$ genau dann zu $\rho<1$, wenn $-1<\alpha<1$, $1<\alpha<3$ ist.

A. Peyerimhoff (Marburg)

12344:

Butzer, P. L. On some theorems of Hardy, Littlewood and Titchmarsh. Math. Ann. **142** (1960/61), 259–269.

The author proves a number of results of which the following theorem is typical. Let $f\in L_p(-\infty, \infty)$, $1<p\leqq 2$. The following three statements are equivalent: $\|f(x+h)-f(x)\|_p=O(h)$ as $h\to 0$; there is a function $g\in L_p(-\infty, \infty)$ such that $ivf^\wedge(v)=g^\wedge(v)$ for almost all v; f is absolutely continuous and $f'\in L_p(-\infty, \infty)$. Theorems of this sort were first established by Hardy and Littlewood (using different methods) [Math. Z. **27** (1928), 565–606; **28** (1928), 612–634].

I. I. Hirschman, Jr. (Erlenbach)

12345:

White, A. J. On the restricted Cesàro summability of double Fourier series. Trans. Amer. Math. Soc. **99** (1961), 308–319.

Let $\phi\in L(0, 0; \pi, \pi)$ and be periodic with period 2π. Let $S[\phi]$ be its Fourier series at $(0, 0)$ and $\sigma_{mn}^{\alpha,\beta}$ the sequence of $(C; \alpha, \beta)$-means of $S[\phi]$. Let $u^a v^b \phi_{a,b}(u, v)$ be the fractional integral of order (a, b):

$$\phi_{a,b}(u, v) = abu^{-a}v^{-b}\int_0^u\int_0^v (u-x)^{a-1}(v-y)^{b-1}\phi(x, y)\,dx\,dy \qquad (a>0, b>0).$$

The sequence $\sigma_{mn}^{\alpha,\beta}\to s(R)$ converges restrictedly to s as $(m, n)\to(\infty, \infty)$ if $(m, n)\to(\infty, \infty)$ so that $(1/\lambda)\leqq mn^{-1}\leqq\lambda$ ($\lambda\geqq 1$), in which case $S[\phi]$ is said to be summable $(C; \alpha, \beta)(R)$ to s. Similarly $\phi(u, v)\to s(C; a, b)(R)$ means that $\phi_{a,b}(u, v)\to s$ as $(u, v)\to(+0, +0)$ so that $(1/\lambda)\leqq uv^{-1}\leqq\lambda$ ($\lambda\geqq 1$). Moore proved that if $\phi(u, v)\to s$ as $(u, v)\to(+0, +0)$, then $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s if $\alpha\geqq 1$, $\beta\geqq 1$ [same Trans. **14** (1913), 73–104]. Here the hypothesis is replaced in the following way (thus extending results of Paley [Proc. Cambridge Philos. Soc. **26** (1930), 173–203] and Bosanquet [Proc. London Math. Soc. (2) **31** (1930), 144–164] on single Fourier series): (i) if for $a\geqq 0$, $b\geqq 0$, $\phi(u, v)\to s(C; a, b)(R)$ and $\phi_{a,b}(u, v)$ is bounded on $(0, 0; \delta, \delta)$ for some $\delta>0$, then $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ to s for $\alpha\geqq 1$, $\beta\geqq 1$, $\alpha>a$, $\beta>b$; (ii) if for $\alpha\geqq 0$, $\beta\geqq 0$, $S[\phi]$ is summable $(C; \alpha, \beta)(R)$ and $\sigma_{mn}^{\alpha,\beta}$ is bounded for m and n sufficiently large, then $\phi(u, v)\to s(C; a, b)(R)$ as $(u, v)\to(+0, +0)$ if $a-2>\alpha$, $b-2>\beta$. A counterexample shows that the boundedness condition on $\phi_{a,b}$ in (i) cannot be entirely removed.

J. Mitchell (University Park, Pa.)

12346:

Korenblyum, B. I. Methods of the theory of functions of a complex variable in generalized harmonic analysis on the line. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 526–531. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Après une brève revue des diverses définitions classiques du spectre d'une fonction bornée, l'auteur donne un aperçu de ses travaux sur la définition du spectre des fonctions g telles que $g(x)=O(e^{a|x|})$ ($a>0$ donné) [Dokl. Akad. Nauk SSSR **111** (1956), 280–282; **115** (1957), 226–229; Trudy Moskov. Mat. Obšč. **7** (1958), 121–148; MR **19**, 46, 968; **21** #266]. Les fonctions g forment un espace de Banach M_a, dual de l'espace L_a des fonctions sommables avec le poids $e^{a|x|}$. Dans la topologie faible de M_a, le sous-espace fermé engendré par g et ses translatées ($g\not\equiv 0$) contient au moins une fonction d'un des types suivants: (1) $e^{-i\lambda x}$ avec $|\mathrm{Im}\,\lambda|\leqq a$; (2) $e^{-i\mu_1 x}\Gamma^{-1}(d+2a\pi^{-1}ix)$; (3) $e^{-i\mu_2 x}\Gamma^{-1}(d-2a\pi^{-1}ix)$ (μ_1, μ_2 réels, $\mathrm{Re}\,d\geqq\frac{1}{2}$). L'auteur définit le spectre harmonique, le spectre anharmonique droit et le spectre anharmonique gauche respectivement comme l'ensemble des λ, des μ_1 et des μ_2. Le spectre anharmonique droit [resp. gauche] contient, en même temps qu'un point μ, tout point à droite de μ [resp. à gauche]. Le spectre harmonique, s'il ne remplit pas toute la bande $|\mathrm{Im}\,\lambda|\leqq a$, est discret. Dans ce cas, c'est l'ensemble des pôles de la fonction méromorphe G transformée de Carleman de g, soit $G(z)=\int_0^\infty g(x)e^{izx}dx$ pour $\mathrm{Im}\,z>a$ et $G(z)=-\int_{-\infty}^0 g(x)e^{izx}dz$ pour $\mathrm{Im}\,z<-a$; la croissance de G sur l'axe réel (dans chaque direction) détermine les spectres anharmoniques de g.

J.-P. Kahane (Montpellier)

12347:

Glicksberg, Irving. A remark on a theorem of Beurling and Helson. Proc. Amer. Math. Soc. **10** (1959), 587.

The theorem of Beurling and Helson [Helson, Ark. Mat. **2** (1953), 475–487; MR **15**, 327; cf. Theorem 2] is as follows:

Let (i) μ be a complex, bounded, regular, Borel measure on a locally compact Abelian group G (under addition),

$$\mu = \lambda + \nu, \quad \lambda = \sum_n a_n \mu_{g_n},$$

where μ_g denotes the measure due to a unit mass at point $g \in G$, and ν is a continuous measure; (ii) $\hat{\mu}$ be the Fourier-Stieltjes transform of μ, i.e., $\hat{\mu}(\alpha) = \int_G \bar{\alpha}(g)d\mu(g)$, $\alpha \in \hat{G}$, the character group of G. Then $|\mu(\alpha)| = 1$ on $\hat{G}$ implies $\sum_n |a_n|^2 = 1$.

The author gives a direct proof resting on four steps. (1) Define μ^* on G by $\mu^*(E) = \bar{\mu}(-E)$, E a Baire subset of G, and observe that $(\mu^*)\hat{} = \hat{\mu}^-$ and therefore from the convolution property that $(\mu * \mu^*)\hat{}(\alpha) = |\hat{\mu}(\alpha)|^2$. (2) By hypothesis, the last term is 1, whence $(\mu * \mu^*)(E) = \mu_0(E)$. (3) The continuous measures form an ideal in the algebra of measures, so that actually $(\lambda * \lambda^*)(E) = \mu_0(E)$. (4) Put $E = \{0\}$ in the last equation, and show by trivial computation that the first term is $\sum_n |a_n|^2$.

This remarkably simple proof does not extend immediately to yield the more potent generalization due to Eberlein [Proc. Amer. Math. Soc. **6** (1955), 310–312; MR **16**, 817; cf. Theorem 1] of a theorem of Wiener. Even so, the reviewer feels that the proof has more than ad hoc significance, and that the ideas employed are capable of yielding other deep results in the field.

P. Masani (Providence, R.I.)

12348:

de Leeuw, K.; Mirkil, H. Translation-invariant function algebras on abelian groups. Bull. Soc. Math. France **88** (1960), 345–370.

Let X be a locally compact, metrizable, abelian group, Y its character group. Let $C_0(X)$ denote the space of continuous complex-valued functions on X vanishing at infinity. Each f in $C_0(X)$ has a Fourier transform, which is a certain measure on Y; the spectrum of f is defined to be the support of its Fourier transform. The spectrum of a subspace of $C_0(X)$ is the union of the spectra of its elements. The authors consider subalgebras A of $C_0(X)$ which are translation-invariant and closed in the sup norm. They show that the spectrum of A is a closed subsemigroup of Y, without isolated points, and "locally a set of multiplicity", meaning there exist functions in A whose transforms are supported in arbitrarily small relatively open subsets. Conversely, every such set is the spectrum of an invariant algebra. Distinct algebras may have the same spectrum; say that spectral synthesis holds for a subset Z of Y if there exists a unique translation-invariant subspace whose spectrum is Z. Call a subsemigroup S of Y "angular" if it contains the zero of Y and is the closure of its interior. Spectral synthesis holds for angular subsemigroups. If A is an invariant algebra with an approximate unit, and $L^q \cap A$ is uniformly dense in A for some finite q, then the spectrum of A is angular. If the spectrum of A is angular, then the space of integrable functions with compact spectra contained in the interior of the spectrum of A is uniformly dense in A. Thus there exists a one-one correspondence between angular subsemigroups of Y and closed invariant algebras A such that A has an approximate unit and a dense integrable subalgebra. The invariant algebras whose spectrum is angular are the algebras of generalized analytic functions of Arsens and Singer [Trans. Amer. Math. Soc. **81** (1956), 379–393; MR **17**, 1226]. If X is the real line, and if A is a proper invariant subalgebra of $C_0(X)$, then A consists precisely of the restrictions to X of all functions continuous on the closed upper (lower) half-plane, including the point at infinity, and analytic in the open upper (lower) half-plane. The authors call A the Phragmen-Lindelöf algebra. If $X = R^n$, and A is an invariant subalgebra of $C_0(X)$, then the spectrum of A is either all of X or is contained in a half-space. The maximal invariant algebras are exactly those with spectrum a half-space, and every proper closed invariant algebra is contained in a maximal one. Every proper closed invariant algebra is anti-symmetric. If $n > 1$, no proper closed invariant subalgebra of $C_0(R^n)$ is invariant under proper motions of R^n. A subalgebra A of $C_0(R^n)$ is invariant under proper conformal linear transformations if and only if it is the intersection of maximal closed invariant algebras.

A. Browder (Providence, R.I.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 12329, 12389, 12392, 12394, 12398.

12349:

Fenyő, István. Eine Bemerkung zur Arbeit von L. Jánossy: On the generalisation of the Laplace-transform in probability theory. Mat. Lapok **10** (1959), 66–71. (Hungarian. Russian and German summaries)

L. Jánossy [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **1** (1951), 343–350; MR **13**, 957] introduced in probability theory a generalization of the Laplace transformation which is based on the eigenfunctions of the integral equations

$$\varphi_s(y) = s \int \varphi_s(x) a(x, y)\, dx,$$

$$\psi_s(x) = s \int \psi_s(y) a(x, y)\, dy.$$

If the kernel $a(x, y) = a(x - y)$, the method yields the Laplace transformation, and if $a(x, y) = a(y/x)$, the Mellin transformation is obtained. In this paper the following generalization of the Laplace transformation is considered. Let V be a class of functions of two variables for which the operation

$$u \overset{x}{*} v = \int_x^y u(x, t) v(t, y)\, dt \qquad (u, v \in V)$$

is commutative. Then there exists a unique similarity transformation T defined on the functions of class V over the interval $[0, \infty)$. For $u \in V$, the generalized Laplace transformation L is given by

$$L(u) = \mathscr{L}[T^{-1}(u)],$$

where $\mathscr{L}$ denotes the Laplace transformation, and T^{-1} is the inverse of T. This transformation has many properties similar to those of the Laplace transformation; for example, $L(u \overset{x}{*} v) = L(u)L(v)$.

A. T. Bharucha-Reid (Eugene, Ore.)

12350:

Gupta, R. K. Certain transformations on unilateral and bilateral operational calculus. Bull. Calcutta Math. Soc. **51** (1959), 191–198.

This paper concerns the two-dimensional Laplace transform. Only absolutely convergent transforms are considered. It is proved that functions satisfying certain stated conditions are Laplace transforms of functions of special forms. As applications some one-dimensional correspondences are found. *Z. Zieleźny* (Wroclaw)

12351:

Agrawal, Bhagwan Das. Some new kernels for the derivation of self-reciprocal functions. II. J. Sci. Res. Banaras Hindu Univ. **8** (1957/58), 243–252.

The author gives 9 examples to Rule 2 in section 9.14 of Titchmarsh's *Fourier integrals* [2nd ed., Clarendon Press, Oxford, 1948]. *A. Erdélyi* (Pasadena, Calif.)

12352:

Agrawal, Bhagwan Das. Some resultant kernels for the derivation of self-reciprocal functions. J. Sci. Res. Banaras Hindu Univ. **8** (1957/58), 253–260.

The author lists 18 kernels which will transform functions which are self-reciprocal for the Hankel transformation of order ν into functions which have the same property for order ν' where $\nu'=\nu$ or $\nu+2$.

A. Erdélyi (Pasadena, Calif.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 12209, 12224, B12611.

12353:

Marčenko, N. V. The existence of solutions of a certain class of non-linear integral equations. Dokl. Akad. Nauk SSSR **137** (1961), 515–518 (Russian); translated as Soviet Math. Dokl. **2**, 305–308.

The author considers the equation (1) $\varphi(x)=\int_0^1 K[x, y, \varphi(y)]dy$ where the function $K(x, y, z)$ is defined either on the set $P(A, B)$ or on P_{AB}, and where $P(A, B)$ is determined by the inequalities $0\leq x\leq 1$, $0\leq y\leq 1$, $A<z<B$, while P_{AB} is given by the inequalities $0\leq x\leq 1$, $0\leq y\leq 1$, $-\infty<A\leq z\leq B<+\infty$. The equation (1) is said to be M-solvable if there exist constants C and D such that if $\varphi(x)$ is any solution of (1) then $C<\varphi(x)<D$. The present work gives several sufficient conditions for the M-solvability of equation (1). The proofs of the theorems are based on ideas given in a work of J. Leray and J. Schauder [Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78]. *H. P. Thielman* (Oxnard, Calif.)

12354:

Wright, E. M. Perturbed functional equations. II. Quart. J. Math. Oxford Ser. (2) **11** (1960), 263–268.

The paper is a continuation of a previous paper of the same title [same Quart. **20** (1949), 155–165; MR **11**, 364] and contains a stronger and more precise form of a result (Lemma 4) of the earlier paper which leads to an improvement and clarification in the deduction of the main theorem of that paper. The new form of the lemma has considerable interest in itself and is stated as follows.

Let I be the interval in which $0\leq x\leq B_1$. Let $M(x, u)=\max_{0\leq\xi\leq x}|u(\xi)|$, $M(u)=M(B_1, u)$. Let S be the set of all absolutely continuous functions $u(x)$ on I for which $u(0)=0$ and $M(u)\leq B_3$. Let $\omega(x)$ be an integrable function of bounded variation in I and let $M(\omega)\leq B_2$. For all x in I and all u in S, let $\chi(x, u)$ be a function of x and a functional of u such that $\chi(x, u)$ is integrable with respect to x and $|\chi(x, 0)|\leq B_4$, $|\chi(x, u_1)-\chi(x, u_2)|\leq B_4M(x, u_1-u_2)$. Then, if $e^{B_1B_2B_3}\leq 1+B_3$, the equation $u(x)=\int_0^x \chi(\xi, u)\omega(x-\xi)d\xi$ has a unique solution $u(x)$ in I and S.

H. R. Pitt (Nottingham)

12355:

Frey, T. Asymptotische Untersuchung eines retardierten Integralgleichungssystems. Period. Polytech. Chem. Engrg. **4** (1960), 233–242.

The functions $y_1(x)$, $y_2(x)$, $\cdots$ are defined as follows: $y_k(x)=0$ if $x<0$, $y_k(x)=1$ if $0\leq x\leq 1$, $y_k(x)$ continuous for $x>0$, and

$$xy_k(x)=\int_0^x y_k(t)\,dt+\int_0^{x-1} y_k(t)\,dt+(k-1)\int_0^{x-1} y_{k-1}(x-1-t)y_1(t)\,dt$$

if $x>1$ (if $k=1$, the last term is zero by definition). The author shows that these conditions can be reduced to $xy_k'(x)=ky_k(x-1)$ with the same initial condition on $-\infty<x<0$ and $0\leq x\leq 1$. Using methods introduced by the reviewer in the case $k=1$ [Publ. Math. Debrecen **1** (1950), 129–134; Nederl. Akad. Wetensch. Proc. **53** (1950), 803–812; MR **12**, 106, 11], the author establishes the asymptotic behaviour of $y_k(x)$ (k fixed, $x\to\infty$). The result is that $y_k(x)$ is approximately equal to a polynomial of degree k, with leading term $e^{-\gamma k}x^k/k!$ (where γ is Euler's constant). The error is very small; it is of the order $x^k/\Gamma(x)$.

{The reviewer remarks that the argument involved in the author's (4.7) is incorrect; it should be replaced by a discussion of what happens if $p\to\infty$. And the reference given to the reviewer's book is completely irrelevant; it should be replaced by those given above.}

N. G. de Bruijn (Eindhoven)

FUNCTIONAL ANALYSIS

See also 12291, 12294, 12340, 12348, 12416.

12356:

Teleman, Silviu. Sur la topologie finie dans les espaces vectoriels. Bull. Sci. Math. (2) **83** (1959), 68–72.

For a linear system X, the finite topology is defined as follows: A subset D is said to be finitely open if the intersection of D with every finite-dimensional subspace is open relative to the usual topology for this finite-dimensional subspace. It was shown in E. Hille and R. S. Phillips, *Functional analysis and semi-groups* [Amer. Math. Soc., Providence, R.I., 1957; MR **19**, 664], that an X endowed with this topology is a Hausdorff space in which addition and scalar multiplication are continuous

in each variable separately. The question was left open as to whether $(x, y) \to x+y$ was continuous. By constructing an ingenious topological basis for X, the author shows in this paper that $x+y$ is continuous in x and y only when X is finite-dimensional. *R. S. Phillips* (Stanford, Calif.)

12357:

Raĭkov, D. A. On a problem of finding an extremal locally convex topology. Moskov. Gos. Ped. Inst. Uč. Zap. **138** (1958), 107–113. (Russian)

12358a:

Bastiani, Andrée. Polyèdres convexes dans les espaces vectoriels topologiques. Séminaire C. Ehresmann, 1957/58, exp. no. 19, 46 pp. Faculté des Sciences de Paris, 1959.

12358b:

Bastiani, Andrée. Cones convexes et pyramides convexes. Ann. Inst. Fourier. Grenoble **9** (1959), 249–292.

Both of these papers contain a detailed exposition (the second with some improvements and extensions) of the material covered in earlier Comptes Rendus notes [C. R. Acad. Sci. Paris **247** (1958), 1943–1946; **248** (1959), 175–178, 2695–2697; MR **21** #2895a, b, 2896].

Victor Klee (Seattle, Wash.)

12359:

Leichtweiss, Kurt. Sur les espaces de Banach autoadjoints. Séminaire C. Ehresmann, 1957/58, exp. no. 4, 33 pp. Faculté des Sciences de Paris, 1959.

The author considers linear spaces with a norm generalized by weakening the condition $\|tx\| = |t|\,\|x\|$ to require only positive homogeneity. Such a space X is called autoadjoint if there is a linear isometry A of the conjugate space X^* onto X.

The bulk of the paper deals with the finite-dimensional cases and the nature of the matrix $\mathfrak{A}$, which is associated to the isometry A in such a space. $\mathfrak{A}$ is called congruent to $\mathfrak{A}_1$ when there exists a non-singular matrix $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B}^*\mathfrak{A}_1\mathfrak{B}$. Theorem 1 says that if $\mathfrak{A}$ is associated with the isometry A of an autoadjoint Minkowski space X, then $\mathfrak{A}$ is congruent to a normal matrix. Theorem 2 gives a standard form of matrix which every autoadjoint space has. Further theorems characterize the Euclidean spaces among the autoadjoint Minkowski spaces. The last part of the paper describes all the autoadjoint spaces of dimension 2. *M. M. Day* (Urbana, Ill.)

12360:

Дэй, М. М. [Day, M. M.]. ★Нормированные линейные пространства [Normed linear spaces]. Translated by A. S. Dynin; edited by D. A. Raĭkov. Biblioteka Sbornika "Matematika". Izdat. Inostr. Lit., Moscow, 1961. 232 pp. 0.78 r.

A translation, with no significant translator's comments, of the author's Ergebnisse monograph [Springer, Berlin, 1958; MR **20** #1187].

12361:

Kreĭn, S. G. On the concept of a normal scale of spaces. Dokl. Akad. Nauk SSSR **132** (1960), 510–513 (Russian); translated as Soviet Math. Dokl. **1**, 586–589.

The author generalizes the concept of analytic scales of spaces developed in his earlier paper [same Dokl. **130** (1960), 491–494; MR **22** #9860], defining a 'normal scale' of spaces. The definition involves a monotone decreasing family of spaces with monotone increasing norm, and a multiplicative inequality between any three of these norms generalizing that valid for the ordinary L_p-spaces. Examples: The L_p-spaces and the H_α-spaces of functions satisfying the scale of Hölder conditions. A convexity theorem is formulated in terms of a notion of maximal normal scale, and applied to the L_p and H_α spaces, and also to the spaces corresponding to the family of norms $\|f\|_\alpha' = (1-\alpha)\int_0^1 t^{-\alpha} f^*(t)\,dt$, $f^*(t)$ being the rearrangement of $f(t)$ in decreasing order. *J. T. Schwartz* (New York)

12362:

Konda, Tomoko. On quasi-normed spaces. III. Proc. Japan Acad. **36** (1960), 189–191.

The author continues his notes on quasi-normed spaces [same Proc. **35** (1959), 340–342, 584–587; MR **22** #5862] with a proof of the closed graph theorem.

M. M. Day (Urbana, Ill.)

12363:

Comfort, W. W.; Gordon, Hugh. Inner product spaces and the tri-spherical intersection property. Proc. Amer. Math. Soc. **12** (1961), 327–329.

A normed linear space E is said to have the tri-spherical intersection property if, for each triple x_1, x_2, x_3 of points of E and each triple ρ_1, ρ_2, ρ_3 of positive numbers such that the three spheres $S_{\rho_i} = \{x : \|x - x_i\| \leq \rho_i\}$ have a common point in E, the spheres have a common point in the smallest flat set containing the x_i. By application of several earlier criteria it is shown that this is another property characterizing inner-product spaces among normed linear spaces. *M. M. Day* (Urbana, Ill.)

12364:

Singer, Ivan. On a theorem of V. Pták concerning best approximation of continuous functions in the metric $\int_a^b |x(t)|\,dt$. Czechoslovak Math. J. **10** (**85**) (1960), 425–431. (Russian summary)

Let $C_{L^1}(a, b)$ be the space of all real-valued continuous functions on $T = [a, b]$ endowed with the norm $|x| = \int_a^b |x(t)|\,dt$. In V. Pták, same J. **8** (**83**) (1958), 267–273, 464 [MR **20** #718; see also S. Ya. Havinson, Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 243–270; MR **21** #254], the following was proved. Let E be an n-dimensional subspace of $C_{L^1}(a, b)$. There exists an $x_0 \in C_{L^1}(T)$ with a non-unique best approximation (by elements of E) if and only if there exist two disjoint sets U_1 and U_2, open in T, and a bounded measurable function $\alpha(t)$ defined on $T - (U_1 \cup U_2)$, with the following properties: (1) $|\alpha(t)| \leq 1$; (2) $\int_{U_1} x(t)dt - \int_{U_2} x(t)dt + \int_{T-(U_1\cup U_2)} x(t)\alpha(t)dt = 0$ for each $x \in E$; (3) there exists a non-zero $x_0 \in E$ vanishing on $T - (U_1 \cup U_2)$. Using extreme points of the adjoint unit sphere, the author proves that α is a convex combination of n functions α_{M_i}, where M_i are independent measurable subsets of $[a, b]$ and $\alpha_M(t) = 1$ for $t \in M$ and $\alpha(t) = -1$ for $t \in [a, b] - M$. *V. Pták* (Prague)

12365:

Singer, Ivan. La meilleure approximation interpolative dans les espaces de Banach. Rev. Math. Pures Appl. **4** (1959), 95–113.

Let E be a real Banach space, x an element of E, $V \subset E$ a linear subspace, $\varphi_1, \varphi_2, \cdots, \varphi_m$ elements of V^* and $c_1, c_2, \cdots, c_m$ a set of real numbers. Then v_0 is an interpolatory element of best approximation for x if (a) $\varphi_i(v_0) = c_i$, $i = 1, 2, \cdots, m$, (b) $\|x - v_0\| = \inf \|x - v\|$ for $v \in V$, $\varphi_i(v) = c_i$, $i = 1, 2, \cdots, m$. The author investigates the problems of existence and uniqueness of such elements and their characterization in particular spaces. A generalization of the original problem is also considered in which the constants depend upon a set of parameters. Of particular interest is the case in which V is finite-dimensional. In this case the elements of V are called polynomials and the theorems involved apply to approximations by polynomials. Classical theorems concerning approximations of continuous functions by polynomials occur as special cases of the author's results.

R. E. Fullerton (College Park, Md.)

12366:

Konstantinesku, Florin [Constantinescu, Florin]. On Čebyšev sets. Dokl. Akad. Nauk SSSR **130** (1960), 21–22 (Russian); translated as Soviet Math. Dokl. **1**, 11–12.

A subset M of a Banach space E is a Čebyšev set if to each x in E there corresponds a unique nearest point in M. It has long been known that if E is smooth and finite-dimensional, then every Čebyšev set M is closed and convex. This was proved anew (for bounded M) by Efimov and Stečkin [same Dokl. **118** (1958), 17–19; MR **20** #1947]; the author notes that boundedness need not be assumed. {It is unknown whether there exists an infinite-dimensional Banach space in which every Čebyšev set is convex. The paper reviewed below goes into this problem in some detail.}

R. R. Phelps (Berkeley, Calif.)

12367:

Klee, Victor. Convexity of Chebyshev sets. Math. Ann. **142** (1960/61), 292–304.

A subset X of a Banach space E is called a Čebyšev set (or Č-set) if each point of E has a unique nearest point in X. It is well known that if E is smooth and finite-dimensional, then E has property P: Every Č-set in E is convex. Whether there exist infinite-dimensional E with property P is an open question. The author proves that in a uniformly convex and uniformly smooth space E, every weakly closed Č-set is convex. (Compare with the result by Efimov and Stečkin [Dokl. Akad. Nauk SSSR **127** (1959), 254–257; MR **21** #5883] that in a uniformly convex and smooth space E, every boundedly compact Č-set is convex.) It is also proved that in a smooth reflexive space E, a Č-set X is convex if the nearest point map of E onto X is (locally) continuous and weakly continuous. Conditions are given which guarantee these continuity properties. A method due to Ficken (unpublished) is used to show that every Hilbert space has property P, provided the following open question has an affirmative answer: Suppose that T is a closed convex subset of the Hilbert space H such that each point of H has a unique farthest point in T. Must T consist of a single point?

R. R. Phelps (Berkeley, Calif.)

12368:

Phelps, R. R. Extreme points of polar convex sets. Proc. Amer. Math. Soc. **12** (1961), 291–296.

This note generalizes a theorem of R. C. Buck [Bull. Amer. Math. Soc. **65** (1959), 130–133; MR **21** #1514] about extreme points of the unit ball of a conjugate space. In this generalization let F and G be dual locally convex linear spaces, let C be a convex subset of F which contains the null-vector $\emptyset$, and let p be the Minkowski functional of C, $p(x) = \inf\{\lambda > 0 : \lambda^{-1}x \in C\}$. Let C^0 be the polar set of C in G, $C^0 = \{y \in G;\ \langle x, y\rangle \leq 1\}$. For y in C^0 let $F_y = \{x : p(x) - \langle x, y\rangle \leq 1\}$. (The subadditivity of p implies that F_y is convex.) Theorem 1: A point y of C^0 is an extreme point of C^0 if and only if the set of differences $F_y - F_y$ is dense in F under the weak topology $\sigma(F, G)$ determined by finite subsets of G.

This is related to Buck's work by showing (Theorem 3) that the density condition is equivalent to equality when $\emptyset$ is an interior point of C; it is applied by considering $C = U$ or U^*, when U is the unit ball of a normed space.

M. M. Day (Urbana, Ill.)

12369:

Welland, R. R. Metrizable Köthe spaces. Proc. Amer. Math. Soc. **11** (1960), 580–587.

Let Λ be a Köthe space and Λ^* its Köthe dual. The author gives lattice-theoretical necessary and sufficient conditions for Λ to be a Banach (or Fréchet) space under the strongest Köthe topology $\beta(\Lambda, \Lambda^*)$, and a sufficient condition for Λ to admit only one Köthe topology. The results are based on the fact that a normal subset A of Λ is $\sigma(\Lambda, \Lambda^*)$ bounded if and only if every nondecreasing sequence of non-negative elements in A is order-bounded in Λ.

I. G. Amemiya (Tokyo)

12370:

Császár, Á. Sur la représentation canonique des fonctionnelles linéaires. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 85–92.

Soit $\mathscr{S}$ une σ-algèbre de parties d'un ensemble H, $H \in \mathscr{S}$, et μ une mesure sur $\mathscr{S}$. On appelle pseudo-fonction par rapport à un anneau $\mathscr{A} \subset \mathscr{S}$, toute famille $g = (g_A)_{A \in \mathscr{A}}$ de fonctions numériques sur H, telle que pour chaque $A \in \mathscr{A}$, g_A soit mesurable $(\mathscr{S})$ sur A, et que $g_{A'}$ et $g_{A''}$ soient égales presque partout sur $A' \cap A''$. Si g est une pseudo-fonction par rapport à un anneau $\mathscr{A} \subset \mathscr{S}$ et si f est une fonction numérique mesurable $(\mathscr{S})$ sur H et nulle en dehors d'un ensemble $A \in \mathscr{A}$, on dit que fg est intégrable sur H si fg_A est intégrable sur A et l'on pose $\int_H fg\, d\mu = \int_A fg_A\, d\mu$. Soit F un espace linéaire de fonctions numériques mesurables $(\mathscr{S})$ sur H tel que $\max(f', f'') \in F$ et $\min(f', f'') \in F$ si $f', f'' \in F$; $\min(f, c) \in F$ si $f \in F$ et $c > 0$; chaque $f \in F$ est nulle en dehors d'un ensemble de mesure σ-finie. Une suite (f_n) de fonctions de F est F-majorée s'il existe $f \in F$ telle que $|f_n(x)| \leq f(x)$ pour tout n et tout $x \in H$. L'auteur démontre le théorème suivant: Une fonctionnelle linéaire T sur F peut s'écrire sous la forme $T(f) = \int_H fg\, d\mu$ pour $f \in F$ où g est une pséudo-fonction par rapport à un σ-anneau $\mathscr{A} \subset \mathscr{S}$, si et seulement si $T(f_n) \to 0$ pour toute suite (f_n) de fonctions de F, F-majorée et convergente vers 0 presque partout sur H. Si $H \in \mathscr{A}$, la pseudo-fonction g peut être remplacée par la fonction g_H. En particulier, on déduit le théorème

de réprésentation intégrale des fonctionnelles continues sur $L^p(H, \mu)$, où g est une pséudo-fonction par rapport à l'anneau des ensembles de mesure σ-finie.

N. Dinculeanu (Bucharest)

12371:

Prasad, S. N. Analogues of some results on sequence spaces for function spaces. Quart. J. Math. Oxford Ser. (2) **11** (1960), 310–320.

The author gives analogues, for function spaces, of some of the results given by Cooke [*Infinite matrices and sequence spaces*, Macmillan, London, 1950; MR **12**, 694]. A set of functions is a function space $\alpha(f)$ if $f, g \in \alpha(f)$ implies $c(f+g) \in \alpha(f)$ for every scalar c. $f \in \alpha^*(f)$ (the dual space of $\alpha(f)$) if $\int_0^\infty |f(x)g(x)|dx < \infty$ for every $g \in \alpha(f)$ (the integral being a Lebesgue integral). Example (as well known): For $\alpha(f) = L_p(0, \infty) = \sigma_p(f)$ we have $\alpha^*(f) = L_{p'}(0, \infty) = \sigma_{p'}(f)$ ($1 \leq p \leq \infty$ and p' the conjugate exponent of p). $f \in \phi(f)$ (called the space of "finite functions") if $f(x) = 0$ in some set $E \subset [0, \infty)$ whose complement CE with respect to $[0, \infty)$ is of finite measure and if f is essentially bounded in CE. $\sigma(f)$ is defined as the space of all functions $f(x)$ such that (i) whenever $g(x) \in \phi(f)$, then $\int_0^\infty |f(x)g(x)|dx \equiv \int_{CE} |f(x)g(x)|dx < \infty$, with the meaning of CE given in the definition of $\phi(f)$; and (ii), whenever $g(x)$ is not in $\phi(f)$, the first of the above integrals does not necessarily have finite value, i.e., does not have finite values for all $f(x)$ in $\sigma(f)$. The author remarks in a footnote: "It seems difficult to give any other (simpler) definition of $\sigma(f)$ which would enable one to prove that $\sigma^*(f) = \phi(f)$ and $\phi^*(f) = \sigma(f)$ from first principles, as in the case of sequence spaces." {The reviewer suggests saying: $\sigma(f)$ $(= \phi^*(f))$ is the set of functions f for which $\int_{CE} |f(x)|dx < \infty$ for every set $CE \subset [0, \infty)$ of finite measure.} Coordinate convergence [cf. loc. cit., p. 283] plays an important role in sequence spaces. The author defines it for functions in the following way: Let $\{f_t(x)\}$ be a family of functions of x defined for all $t \geq 0$ where t is a parameter. If, for almost all $x \geq 0$, $\lim_{t\to\infty} f_t(x) < \infty$ exists, then $\{f_t(x)\}$ is said to be coordinate convergent (c-cgt). If this limit exists uniformly in x for almost all $x \geq 0$, then $\{f_t(x)\}$ is called parametric convergent (t-cgt). Projective convergence is defined in the following way: Let $\beta(f) \subset \alpha^*(f)$ and $F_g(t) = \int_0^\infty f_t(x)g(x)dx$, where $f_t(x) \in \alpha(f)$ and $g(x) \in \beta(f)$, then $\{f_t(x)\}$ is projective convergent relative to $\beta(f)$ or $\alpha(f)\beta(f)$-cgt if $\lim_{t\to\infty} F_g(t) < \infty$ exists for every $g \in \beta(f)$. The author remarks that, unlike the case of sequence spaces, $\phi(f) \subset \beta(f)$ does not ensure that projective convergence implies coordinate convergence. {Remark of the reviewer: If we define $\{f_t(x)\}$ to be c-cgt if $\lim_{t\to\infty} \int_e f_t(x)dx$ exists for every set $e \subset [0, \infty)$ of finite measure, and to be t-cgt if this limit exists uniformly in e for all sets $e \subset [0, \infty)$ of measure ≤ 1, then c-cgt would be defined in a manner more analogous to the case of sequence spaces and $\phi(f) \subset \beta(f)$ would ensure that projective convergence implies coordinate convergence.} $\alpha(f)$ is said to be normal if, whenever $f(x) \in \alpha(f)$ and $|g(x)| \leq |f(x)|$ for almost all $x \geq 0$, $g(x) \in \alpha(f)$. $f_t(x) \in \mathcal{M}(f)$ when $f_t(x)$ is either monotonic increasing or monotonic decreasing with respect to t in $[0, \infty)$ for almost all $x \geq 0$. From the various statements may be mentioned the following: (i) If $f_t(x) \in \mathcal{M}(f)$ and is $\alpha(f)\beta(f)$-cgt and $\beta(f) \supset \phi(f)$, then $f_t(x)$ is t-cgt. (ii) When $f_t(x)$ is t-cgt, $\beta(f)$ is normal and $\beta(f) \subset \sigma_\infty(f)$, then $f_t(x)$ is $\alpha(f)\beta(f)$-cgt if and only if to every $g(x) \in \beta(f)$ and every $\varepsilon > 0$ there corresponds a number $T(\varepsilon, g)$ such that for all $t, t' \geq T(\varepsilon, g)$, $\int_0^\infty |g(x)\{f_t(x) - f_{t'}(x)\}|dx \leq \varepsilon$.

G. Goes (Evanston, Ill.)

12372:

Prasad, S. N. Function to sequence mapping. Quart. J. Math. Oxford Ser. (2) **12** (1961), 45–51.

The author considers transformations of the type $u_n(g) = \int_0^\infty F_n(x)g(x)dx$ where the integral is a Lebesgue integral. $F_n(x)$ is called the transformer. For the notations we refer to the review of the foregoing paper of the author [#12371]. $F_n(x) \in \sigma_1(f)$ is said to be convergent in mean if $\lim_{m>n\to\infty} \int_0^\infty |F_n(x) - F_m(x)|dx = 0$. A set $X \subset \sigma_1(f)$ is said to be integral-bounded (int-bd) if $\sup \int_0^\infty |g(x)|dx \leq M$ $(g \in X)$ for some constant M. A set $X \subset \sigma_\infty(f)$ is said to be collectively bounded, when $\sup |f(x)| \leq M$ $(f \in X)$ for almost all $x \geq 0$ and for some constant M. A set $X \subset \Gamma$ (the space of convergent sequences) is collectively convergent when, with $u = \{u_n\}$, $\lim \sup_{m>n, u\in X} |u_n - u_m| = 0$. The following statements are made: (i) If $F_n(x) \geq 0$ for almost all $x = 0$ and for every n, then $F_n(x)$ transforms $\alpha(f) \subset \sigma_\infty(f)$ into a normal sequence space β if and only if $F_n(x) \in \sigma_1(f)$ and $\int_0^\infty F_n(x)dx \in \beta$, where $\alpha(f)$ contains the constant functions. (ii) If $\phi(f) \subset \beta(f) \subset \sigma_1(f)$ and $\{F_n(x)\} \in \mathcal{M}(f)$, then $F_n(x)$ transforms $\beta(f)$ into Γ if and only if $F_n(x) \in \beta^*(f)$ and $\{F_n(x)\}$ is parametric convergent. (iii) $F_n(x) \in \sigma_1(f)$ transforms collectively bounded sets in $\sigma_\infty(f)$ into collectively convergent sets in Γ, if and only if $F_n(x)$ is convergent in mean. (iv) $F_n(x) \in \sigma_\infty(f)$ transforms int-bd sets in $\sigma_1(f)$ into collectively convergent sets in Γ if and only if $\{F_n(x)\}$ is parametric convergent.

G. Goes (Evanston, Ill.)

12373:

Preisendorfer, Rudolph W.; Roos, Bernard W. Recurrence-partitions of finite measure spaces with applications to ergodic theory. Trans. Amer. Math. Soc. **99** (1961), 91–101.

From the authors' summary: "The object of this note is to develop and apply a certain class of measure-theoretic partitions induced by invertible measure-preserving transformations on finite measure spaces. The utility of such partitions is illustrated by the following examples which are considered, with others, in detail below: (a) a generalization of Kac's theorem on the integral of the recurrence function over a measurable set; (b) some integral theorems for the exit function, the natural complement of the recurrence function; and (c) a special criterion for the ergodicity of invertible measure-preserving transformations on finite measure spaces which may conceivably be of physical interest."

R. V. Chacon (Ithaca, N.Y.)

12374:

Golovkin, K. K. On imbedding theorems. Dokl. Akad. Nauk SSSR **134** (1960), 19–22 (Russian); translated as Soviet Math. Dokl. **1**, 998–1000.

The author considers types of proofs of theorems on imbedding spaces of functions with bounded domains of finiteness with norms of $W_p^{l,k}$ and $W_p^{(l)}$ as given by Il'in [same Dokl. **129** (1959), 1214–1217; MR **22** #2882], and discusses papers of Gagliardo [Ricerche Mat. **7** (1958), 102–137; MR **21** #1526] and Nirenberg [Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162; MR **22** #823].

A theorem is proven relating the norms in $L_q(E_s \cap \Omega)$, $W_p^{l,n-s}(\Omega)$ and $L_{p(lq-n+s)/(lp-n+s)}$, and shown to be equivalent to a theorem of Sobolev on the imbedding of $W_p^{(l)}(E_n)$ in $L_q(E_s)$. Here Ω is the domain of finiteness in E_n, and l is the differential order of the norm.

S. Hoffman (Hartford, Conn.)

12375:

Byhovskiĭ, È. B.; Smirnov, N. V. Orthogonal decomposition of the space of vector functions square-summable on a given domain, and the operators of vector analysis. Trudy Mat. Inst. Steklov. **59** (1960), 5–36. (Russian)

Soit $\Omega \subset E_3$ un domaine borné dont la frontière S est homéomorphe à une sphère. On utilise les notations suivantes: G [resp. $\mathring{G}$] est l'adhérence dans $L_2(\Omega)$ de l'espace des vecteurs $v = \operatorname{grad} \varphi$, où φ est à dérivée continue [resp. φ' est continue et $\varphi|S = 0$]; J [resp. $\mathring{J}$] est l'adhérence dans $L_2(\Omega)$ de l'espace des vecteurs v à dérivée continue, tels que $\operatorname{div} v = 0$ [resp. $\operatorname{div} v = 0$ et $v_n|S = 0$]; U est l'adhérence dans $L_2(\Omega)$ de l'espace des vecteurs $v = \operatorname{grad} h$, où h est une fonction harmonique. Par v_n et v_τ on désigne les composantes normale et tangentielles à S de v. L'espace $W_2^n(\Omega)$ de Sobolev est formé des fonctions qui sont de puissance seconde sommable, aussi bien que leurs dérivées jusqu'à l'ordre n. Les auteurs étudient la décomposition de L_2 en somme directe orthogonale des espaces de plus haut. Parmi les plus importants, citons les résultats suivants: (1) $L_2 = \mathring{G} + U + \mathring{J} = \mathring{G} + J = G + \mathring{J}$; (2) pour tout $u \in \mathring{J}$ [resp. $u \in J$] il existe $v \in W_2^1$ tel que $u = \operatorname{rot} v$; en outre, on a $\operatorname{div} v = 0$, $v_\tau|S = 0$ [resp. $v_n|S = 0$] et $\|v\|_{W_2^1} \leq c\|u\|_{L_2}$; (3) si $u \in W_2^n$ alors la projection Pu de u dans un des espaces $\mathring{G}$, J, U, $\mathring{J}$, G appartient encore à W_2^n et $\|Pu\|_{W_2^1} \leq c\|u\|_{W_2^n}$. Une étude analogue a été faite par H. Weyl [Duke Math. J. **7** (1940), 411–444; MR **2**, 202]. Certains résultats ont été obténus antérieurement par K. Friedrichs [Comm. Pure Appl. Math. **8** (1955), 551–590; MR **19**, 407]. Les auteurs considèrent aussi le cas des domaines Ω multiplement connexes, bornés ou non. *N. Dinculeanu* (Bucharest)

12376:

Morel, Henri. Existence de noyaux sur $R \times R$, indéfiniment différentiables dans l'ouvert $\{(x, y) \in R \times R, x \neq y\}$, semi-régulier en x, non semi-régulier en y. Ann. Inst. Fourier. Grenoble **10** (1960), 303–306.

Let $\mathscr{E}$ denote the space of infinitely differentiable functions on R and let $\mathscr{D}$ be its subspace consisting of functions with compact carriers. A distribution $T_{x,y}$ on $R \times R$ is said to be semiregular in x [resp. semiregular in y] if $\varphi \to T \cdot \varphi$ maps $\mathscr{D}_y$ into $\mathscr{E}_x$ [resp. $\mathscr{D}_x$ into $\mathscr{E}_y$].

The author constructs a locally integrable function $g(x, y)$ on $R \times R$, and therefore a distribution, which is infinitely differentiable outside the origin, semiregular in x, and not semiregular in y. *Z. Zieleźny* (Wroclaw)

12377:

Roumieu, Charles. Sur quelques extensions de la notion de distribution. Ann. Sci. École Norm. Sup. (3) **77** (1960), 41–121.

C'est une étude de certaines généralisations des distributions de Schwartz sur la droite numérique, obtenues en remplaçant l'espace fondamental $\mathscr{D}$ (des fonctions indéfiniment dérivables à support compact) par un d'une certaine classe, assez vaste, d'espaces fondamentaux.

Chapitre I. On considère l'espace fondamental $\mathscr{D}(\{M_p\})$ des fonctions $f \in \mathscr{D}$ vérifiant $|f^{(p)}(x)| \leq Ah^p M_p$ ($p = 0, 1, 2, \cdots$), pour un A, un h (qui dépendent de f), muni d'une topologie naturelle. On suppose (sans restreindre la généralité) que la suite M_p soit logarithmiquement convexe; on suppose que $\mathscr{D}(\{M_p\})$ contienne des fonctions non identiquement nulles (ce qui équivaut à $\sum M_{p-1}/M_p < \infty$, ainsi qu'à $\sum M_p^{-1/p} < \infty$); la classe des $\{M_p\}$ vérifiant ces conditions est notée $\mathfrak{M}$. Le dual $\mathscr{D}'(\{M_p\})$ est l'espace correspondant de distributions généralisées (si $M_p < \infty$, on identifie l'espace $\mathscr{D}'$ des distributions ordinaires avec un sous-espace de $\mathscr{D}'(\{M_p\})$; si $M_p = \infty$ pour $p > r$, $\mathscr{D}'(\{M_p\})$ est l'espace des distributions ordinaires d'ordre $\leq r$). On définit support, dérivation, multiplication et convolution comme dans le cas des distributions ordinaires et on établit des propriétés usuelles. La dérivation applique $\mathscr{D}'(\{M_p\})$ dans $\mathscr{D}'(\{M_{p-1}\})$ (dans le même $\mathscr{D}'(\{M_p\})$, si $M_{p+1}^{1/p} = O(M_p^{1/p})$). La convolution applique $\mathscr{D}'(\{M_p\}) \times \mathscr{E}'(\{M_p'\})$ dans $\mathscr{D}'(\{Q_p\})$, où $\mathscr{E}'(\{M_p'\})$ est le sous-espace de $\mathscr{D}'(\{M_p'\})$ d'éléments à support compact, et $Q_p = \inf_{q \leq p} M_q M'_{p-q}$. Chaque distribution généralisée est de la forme $\sum_0^\infty D^p \mu_p$, où μ_p sont des mésures. $\mathscr{D}(\{M_p'\})$ est dense dans $\mathscr{D}(\{M_p\})$ et on identifie $\mathscr{D}'(\{M_p\})$ avec un sous-espace de $\mathscr{D}'(\{M_p'\})$, pourvu que $M_p'^{1/p} = O(M_p^{1/p})$; après ces identifications $\bigcup_{(M_p) \in \mathfrak{M}} \mathscr{D}'(\{M_p\})$ est un espace vectoriel. Il y a beaucoup d'autres propriétés.

Chapitre II. On considère l'espace fondamental $\mathfrak{S}_\alpha(\{M_p\}, \{N_p\})$ des fonctions indéfiniment dérivables vérifiant $\|x^q f^{(p)}(x)\|_\alpha \leq Ah^q k^p M_p N_q$ pour un A, un k, un h (qui dépendent de f), muni d'une topologie naturelle; $\|\ \|_\alpha$ désigne la norme dans L^α, $\{M_p\}$, $\{N_p\}$ sont supposées logarithmiquement convexes. On a $\mathfrak{S}_\infty(\{M_p\}, a^p) = \mathscr{D}(\{M_p\})$ lorsque $\{M_p\} \in \mathfrak{M}$. $\mathfrak{H}_\alpha(\{N_p\}) = \mathfrak{S}_\alpha(\{p!\}, \{N_p\})$ est l'espace des fonctions analytiques f dont les extensions sont holomorphes et vérifient $\|x^q f(x + iy)\|_\alpha \leq Ak^q N_q$ pour $|y| \leq 1/h$, pour un A, un k, un h (qui dépendent de f); il ne se réduit pas à (0) si et seulement si $\sum \exp(-aN_p/N_{p-1}) < \infty$ pour un a. On suppose que $\liminf M_p/pM_{p-1} > 0$, ce qui entraîne que $\mathfrak{H}_\alpha(\{N_p\})$ est un sous-espace de $\mathfrak{S}_\alpha(\{M_p\}, \{N_p\})$, dense pourvu qu'il ne se réduise pas à (0), et on peut identifier $\mathfrak{S}'(\{M_p\}, \{N_p\})$ avec un sous-espace de $\mathfrak{H}_\alpha'(\{N_p\})$. Chaque distribution généralisée de $\mathfrak{S}_\alpha'(\{M_p\}, \{N_p\})$ est de la forme $\sum_0^\infty D^p g_p$, où $g_p \in L^\beta$, $1/\alpha + 1/\beta = 1$ (g_p est une mesure si $\alpha = \infty$). Dans certaines conditions sur $\{N_p\}$ chaque élément de $\mathfrak{H}_\alpha'(\{N_p\})$ est de la forme $\lim_{y \to 0+} \varphi(x + iy) - \varphi(x - iy)$ où φ est holomorphe pour $y \neq 0$; φ est déterminée à un terme fonction entière près; ceci permet de définir le support (comme le complémentaire de l'ouvert des points sur la droite où φ se prolonge holomorphiquement) et la définition coïncide avec l'antérieure [cf. aussi M. Sato, J. Fac. Sci. Univ. Tokyo Sect. I. **8** (1959), 139–193; MR **22** #4951]. On définit multiplication et convolution et on établit des propriétés usuelles. On étend quelques résultats à un cas où $\mathfrak{H}_\alpha(\{N_p\})$ est remplacé par un espace de fonctions holomorphes dans un voisinage de l'axe réel qui ne contient généralement pas une bande rectiligne.

Chapitre III. La transformation de Fourier applique isomorphiquement $\mathfrak{S}_2'(\{M_p\}, \{N_p\})$ sur $\mathfrak{S}_2'(\{N_p\}, \{M_p\})$; quelques propriétés de la croissance d'une transformée (par exemple, le théorème de Paley-Wiener pour $\mathfrak{S}_2'(\{p!\}, \{p!\})$) sont établies. Dans certaines conditions sur

$\{M_p\}$ il existe une distribution généralisée γ de support l'origine telle que $f \to \gamma * f$ applique un espace de fonctions indéfiniment dérivables [resp. un espace de fonctions analytiques] sur l'espace des distributions généralisées de $\mathscr{D}'(\{M_p\})$ à support limité à gauche [resp. sur $\mathscr{E}_\infty'(\{M_p\}, \{N_p\})$]. Le segment-support de $\varphi * \psi$ est la somme des segments-supports de φ, $\psi \in \bigcup_{(M_p) \in \mathfrak{M}} \mathscr{D}'(\{M_p\})$ à supports compacts; mais dans certaines conditions sur $\{M_p\}$ l'espace $\mathscr{E}_2'(\{M_p\}, \{+\infty\})$ contient un élément γ de segment-support (a, b), $-\infty < a < b < \infty$, tel que $\gamma(x) * \gamma(-x)$ est de support l'origine. Dans certaines hypothèses chaque $\varphi \in \mathscr{D}'(\{Q_p\})$ à support compact est de la forme $\sum_0^\infty D^p \mu_p$, les mesures μ_p ayant leurs supports contenus dans le segment-support de φ, mais il existe des distributions généralisées de support l'origine qui ne sont pas égales à une somme $\sum_0^\infty c_p \delta^{(p)}$ convergente dans un $\mathscr{D}'(\{M_p\})$. Il y a des théorèmes sur les équations de convolution, par exemple, une condition nécessaire et suffisante sur la transformée de Fourier d'un $\varphi \in \bigcup_{(M_p) \in \mathfrak{M}} \mathscr{E}_\infty'(\{M_p\}, \{p!\})$ vérifiant $\varphi = 0$ pour $x < 0$, pour que l'equation $\varphi * \rho = \delta$ admette une solution $\rho \in \bigcup_{(M_p) \in \mathfrak{M}} \mathscr{D}'(\{M_p\})$ à support limité à gauche.

S. Łojasiewicz (Kraków)

12378:

Mikusiński, J. Distributions à valeurs dans les réunions d'espaces de Banach. Studia Math. **19** (1960), 251–285.

For normed spaces X_1, X_2, write $X_1 \ll X_2$ if $X_1 \subset X_2$ and $|x|_{X_2}/|x|_{X_1}$ is bounded in X_1. Let (X) be a family of normed spaces forming a directed set for $\ll$; their union is written $\mathscr{X}$ with convergence defined for a sequence as membership and convergence in some particular subspace. Continuity, differentiation and integration of functions from a real interval to this union are defined. A union $\mathscr{X}$ of spaces (X) is called completable if $X_1 \ll X_2$ implies that $\overline{X}_1 \ll \overline{X}_2$; here bars denote completions. $\overline{\mathscr{X}}$ is the union of the $(\overline{X})$. A similar process of forming unions is applied to directed sets of unions (X). Quotients of unions with respect to linear spaces are defined and studied. If $\mathscr{X}$ is the union of (X), $Y_{k+1} \subset Y_k$ a sequence of subspaces of the X, then a space $\mathscr{X}(Y_k)$ whose elements are residue classes of points of the X modulo the Y_k, and in which convergence is defined as convergence of the residue classes for all fixed k, is defined and shown to be a union of normed spaces.

The purpose of these constructions is to unify the theory of distributions and Mikusiński operators and to define distributions to the space of Mikusiński operators. The space of distributions of finite order with values in a Banach space X is defined as the completion of the union of the quotients C/P_q, where C is the space of continuous functions to X, P_q the space of polynomials of order $< q$ with coefficients in X. Distributions of finite order in a space of unions of Banach spaces is defined as the union of the corresponding spaces of distributions to the Banach spaces. If $\mathscr{D}$ is the space of distributions to a union $\mathscr{X}$, Y_k the space of distributions zero on $(-k, k)$, then $\mathscr{X}(Y_k)$ is the space of distributions localized on $(-k, k)$, and the union for $k = 1, 2, 3, \cdots$ of these spaces is the space of distributions of arbitrary order to $\mathscr{X}$.

Consideration of the space of distributions to the space of Mikusiński operators (which is not a topological space) is used to discuss certain differential equations.

J. L. B. Cooper (Cardiff)

12379:

Nagasawa, Masao. Isomorphisms between commutative Banach algebras with an application to rings of analytic functions. Kōdai Math. Sem. Rep. **11** (1959), 182–188.

Let X, Y be compact Hausdorff spaces. Let A, B be closed (in general not self-adjoint) subalgebras with unit of $C(X)$, $C(Y)$, respectively. The main theorem of the paper states that the algebras A and B are isomorphic (no topology) if and only if the Banach spaces A and B are isometric. The proof actually shows that any isometry T of A onto B is of the form $Tf = g(Mf)$, where M is an algebra-isomorphism and $g \in B$ has modulus identically 1 on the maximal ideal space of B (which in general contains Y as a proper subset).

{Reviewer's comment: The theorem is false without units. For instance, let A be the functions analytic in the disk $|z| < 1$ with continuous boundary values, and let B consist of all $f \in A$ that vanish at the origin. The algebras A and B are not isomorphic, since B has no unit. But $f \to zf$ is an isometry of A onto B.}

Let Ω be a plane domain, maximal in the sense of Kakutani and Rudin [Kakutani, *Lectures on functions of a complex variable*, pp. 71–83, Univ. of Michigan, Ann Arbor, Mich., 1955; MR **16**, 1125; Rudin, Trans. Amer. Math. Soc. **78** (1955), 333–342; MR **16**, 685]. Let $H^\infty(\Omega)$ be the bounded analytic functions in Ω. It is known that the algebra $H^\infty(\Omega)$ characterizes Ω conformally. Hence the theorem of the present paper shows that the Banach space $H^\infty(\Omega)$ also characterizes Ω conformally.

H. Mirkil (Hanover, N.H.)

12380:

deLeeuw, Karel; Rudin, Walter; Wermer, John. The isometries of some function spaces. Proc. Amer. Math. Soc. **11** (1960), 694–698.

Let Z be the disk $|z| < 1$ in the complex plane. Let W be the circle $|w| = 1$. For f defined in Z, and $0 < r < 1$, let f_r be defined on W by $f_r(w) = f(rw)$. An f analytic in Z belongs to the space $H^p(Z)$ if and only if f_r is bounded in $L^p(W)$ as $r \to 1$. The authors find all isometries of $H^1(Z)$ and all isometries of $H^\infty(Z)$. {The reviewer believes that the isometry problem for $H^p(Z)$ is still open when $p \neq 1, 2, \infty$.}

An appended note acknowledges that the H^∞ case is contained in the paper of Nagasawa reviewed above. {The twice-proved theorems are wrongly cited, however.}

H. Mirkil (Hanover, N.H.)

12381:

Honda, Koji. A characteristic property of L_p-spaces ($p > 1$). II. Proc. Japan Acad. **36** (1960), 123–127.

[For part I, by the author and S. Yamamuro, see same Proc. **35** (1959), 446–448; MR **22** #2894.] The author obtains a characterization of L_p spaces by conjugately similar spaces [H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR **12**, 420]. Let R be a conjugately similar space with a conjugately similar transformation T to the conjugate space $\bar{R}$. Then, putting $m(x) = \int_0^1 T\xi x(x) d\xi$, we obtain a modular m on R. The author proves that for the second norm $\|x\|$ on R and the first norm $\|\bar{x}\|$ on $\bar{R}$ by this modular m, if $\|x\| = \|y\|$ implies $\|Tx\| = \|Ty\|$, then R is a L_p space by m for some $p > 1$.

H. Nakano (Kingston, Ont.)

12382:

Konyuškov, A. A. Some classes of functions. I. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 841–870. (Russian)

Soit $\varphi(t)$ une fonction positive sur $(0, 2\pi)$, et soient $1 \leq p \leq \infty$, k un entier positif, $M > 0$. Soit, enfin, $\Delta_t^{(k)} f$ la différence symétrique d'ordre k de f:

$$\Delta_t^{(k)} f = \sum_{0 \leq m \leq k} (-1)^m \binom{k}{m} f[x + (k-2m)t].$$

$H_{\varphi,k,p}^{M}$ est la classe des fonctions $f \in L_p(0, 2\pi)$ si $1 \leq p < \infty$, et celle des fonctions $f \in C_{2\pi}$ si $p = \infty$, pour lesquelles $\|\Delta_t^{(k)} f\|_p / \varphi(t) \leq M$. $H_{\varphi,k,p}$ est la classe des fonctions $f \in L_p$ pour lesquelles $\limsup \|\Delta_t^{(k)} f\|_p / \varphi(t) < \infty$; $H_{\varphi,k,p}^{\infty}$ est l'ensemble de toutes les fonctions de L_p qui n'appartiennent pas à $H_{\varphi,k,p}$, si $1 \leq p < \infty$ (et de $C_{2\pi}$ qui n'appartiennent pas à $H_{\varphi,k,\infty}$). Voici quelques faits démontrés par l'auteur: Pour que $H_{\varphi,k,p}^{\infty} \neq \emptyset$ il faut et il suffit que $\liminf_{t \to +0} \varphi(t) = 0$. Pour que $H_{\varphi,k,p}$ contienne des fonctions non équivalentes à une constante, il faut et il suffit que $\limsup t^k / \varphi(t) < \infty$. Si $H_{\varphi,k,p}$ ne coïncide pas avec L_p, cette classe est un sous-ensemble de première catégorie dans L_p. *S. Mandelbrojt* (Paris)

12383:

Konyuškov, A. A. Some classes of functions. II. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 135–155. (Russian)

Résultats semblables à ceux démontrés dans la première partie [voir l'analyse précédente]. *S. Mandelbrojt* (Paris)

12384:

Wang, Sheng-Wang. A note on the Marcinkiewicz-Orlicz space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 707–710. (Russian summary, unbound insert)

Using the same notation as in J. Albrycht [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 1–3; MR **17**, 953], the author studies the spaces L_{SM} and L_{SM}^{*}. Elements of L_{SM} of absolutely continuous norm are defined. It is shown how the sphere of radius one about the set of these elements is related to all of L_{SM}. *M. M. Day* (Urbana, Ill.)

12385:

Hill, C. K. On the singly-infinite Hilbert matrix. J. London Math. Soc. **35** (1960), 17–29.

Let $A_0^{\infty}(\lambda) = (m+n+\lambda)^{-1}$, where $m, n = 0, 1, 2, \cdots$, and λ is complex $\neq 0, -1, -2, \cdots$. It is shown that for all such λ, every complex number with a positive real part is a latent root of $A_0^{\infty}(\lambda)$, thus generalizing a result of Rosenblum [Proc. Amer. Math. Soc. **9** (1958), 137–140; MR **20** #1139] when λ is real and positive. It is also shown that if λ is real the only other latent root is $-\pi|\operatorname{cosec} \pi\lambda|$ when $\lambda < 0$. Also the multiplicities of all latent roots for λ real are found. *C. R. Putnam* (Lafayette, Ind.)

12386:

Calderón, Alberto; Spitzer, Frank; Widom, Harold. Inversion of Toeplitz matrices. Illinois J. Math. **3** (1959), 490–498.

Let

$$f \sim \sum_{-\infty}^{+\infty} f_k e^{ikx},$$

T_f the Toeplitz matrix $(f_{j-k})_{j,k=0,1\cdots}$, and l_p^{+} the space of sequences $(x_0, x_1, \cdots)$ with the usual p norm. For a continuous $f(\theta) \neq 0$ on $[-\pi, \pi]$, let $I = (2\pi)^{-1} \int_{-\pi}^{\pi} (\arg f)$. It is shown that if $\varphi(\theta) \sim \sum_{-\infty}^{+\infty} \varphi_k e^{ik\theta} \neq 0$ and $\sum |\varphi_k| < \infty$, then T_{φ}, as an operator from l_{∞}^{+} to l_{∞}^{+}, is invertible if $I(\varphi) = 0$; is one-one and has range a subspace of deficiency $I(\varphi)$, if $I(\varphi) > 0$; is onto and has null space of dimension $-I(\varphi)$, if $I(\varphi) < 0$. This type of result was found independently by M. G. Kreĭn [Uspehi Mat. Nauk **13** (1958), no. 5, 3–120; MR **21** #1507]. Also, if $\varphi(\theta) \sim \sum_{-\infty}^{+\infty} \varphi_k e^{ik\theta} \in L_{\infty}(-\pi, \pi)$, $1/\varphi \in L_{\infty}(-\pi, \pi)$, and there exists an $\arg \varphi \in L_2(-\pi, \pi)$ having a conjugate function of class $L_{\infty}(-\pi, \pi)$, then T_{φ}, as an operator from l_2^{+} to l_2^{+}, is invertible. In both of these theorems, formulae for T_{φ}^{-1} are given (when T_{φ} is invertible). The last theorem has as consequences a result of Wintner [Math. Z. **30** (1929), 228–282; p. 279] on triangular T_{φ}, and a result of Hartman and Wintner [Amer. J. Math. **76** (1954), 867–882; MR **17**, 499] on Hermitian T_{φ}.

P. Hartman (Baltimore, Md.)

12387:

Sarymsakov, T. A. On the question of existence and uniqueness of solutions of a class of linear operator equations. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1959**, no. 1, 69–74. (Russian. Uzbek summary)

Let a linear, continuous operator A, acting in a linear topological space E, have the following properties: (1) there exists a bounded, closed, partially ordered set Q such that $AQ \subseteq Q$; (2) for arbitrary $x, y \in Q$ there exist an integer m, a number λ $(0 < \lambda < 1)$ and elements $z, u, v \in Q$ such that $z \leq A^m x$, $z \leq A^m y$, $A^m x - z = (1-\lambda)u$, $A^m y - z = (1-\lambda)v$; (3) there exists a set $Q_0 \subset Q$ such that $\theta \notin Q_0$, $AQ_0 \subseteq Q_0$, $f(x) = 1$ $[\neq 1]$ for $x \in Q_0$ $[x \notin Q_0]$, where f is a continuous functional. It is proved that, under these conditions on Q_0, there exists a unique solution of the equation $Ax = x$.

By specializing the hypotheses, the author formulates a theorem containing all the results obtained earlier which relate to the question of the regularity of homogeneous Markov-processes, i.e., to the question of the existence of a final probability distribution.

S. N. Kračkovskiĭ (RŽMat **1960** #6694)

12388:

Putnam, C. R. On square roots and logarithms of self-adjoint operators. Proc. Glasgow Math. Assoc. **4**, 1–2 (1958).

In his earlier paper [Proc. Amer. Math. Soc. **8** (1957), 768–769; MR **19**, 565] the author has proved that if a bounded operator T on a Hilbert space has the property that the numerical range $W(T)$ of T does not contain the origin as an interior point and if T^2 is a normal operator, then T is normal itself. In this note he uses this to prove that if T^2 is a non-negative selfadjoint operator and $W(T)$ belongs to the closed right half-plane, then T is the unique square root of T^2 and is therefore a non-negative self-adjoint operator. Furthermore, if (1) T is bounded, (2) $\exp T = \int \lambda \, dE$ is a positive definite selfadjoint operator, and (3) $\|T\| \leq 2 \log 2$, then $T = \int \log \lambda \, dE$ ($\log \lambda$ real) is a selfadjoint operator. *S. Kurepa* (College Park, Md.)

12389:

Hörmander, Lars. Estimates for translation invariant operators in L^p spaces. Acta Math. **104** (1960), 93–140.

Let $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$ be points in R^n, n-dimensional Euclidean space, and let $x+y=(x_1+y_1, \cdots, x_n+y_n)$, $x\cdot y=x_1y_1+\cdots+x_ny_n$, $|x|=(x\cdot x)^{1/2}$. For $u(x)$, defined for $x\in R^n$, let $(\tau_h u)(x)=u(x-h)$. Let L^p be the space of measurable functions on R^n with integrable pth power, and write $\|u\|_p=\int |u|^p dx$. A bounded linear operator A from L^p to L^q such that $\tau_h A=A\tau_h$ for all $h\in R^n$ is said to be translation-invariant. The present paper is in part a survey of translation-invariant operators, and in part a reworking and very considerable extension of this theory.

The paper has three sections. Part I is devoted to generalities concerning translation-invariant operators. Let $\mathscr{S}$ be the space of infinitely differentiable functions on R^n vanishing at infinity more rapidly than any polynomial, and let $\mathscr{S}'$ be its dual, the space of tempered distributions. The author notes that to each translation-invariant operator A from L^p to L^q there may be associated a unique distribution $T\in\mathscr{S}'$ such that $Au=T*u$ for $u\in\mathscr{S}$. The space of distributions $T\in\mathscr{S}'$ such that $\|T*u\|_q\leq C\|u\|_p$ for all $u\in\mathscr{S}$ is denoted by L_p^q. The set of Fourier transforms $T^\wedge$ of T's in L_p^q is denoted by M_p^q. The elements of M_p^q are called multipliers of type p-q, and for $u\in\mathscr{S}$ $T*u=\mathscr{F}^{-1}(T^\wedge\mathscr{F}u)$, where $\mathscr{F}$ denotes the Fourier transform $\mathscr{F}u=u^\wedge$. To each $T\in L_p^q$ and $T^\wedge\in M_p^q$ may be assigned the norm of the corresponding transformation of L^p into L^q. It is easy to verify the following: $M_p^p\subset L^\infty$; $M_p^q\subset L^q_{\text{loc}}$ if $p\geq 2$; and $M_p^q\subset L^p_{\text{loc}}$ if $q\leq 2$. On the other hand, it is shown that if $p<2<q$, then there exist elements in M_p^q which are distributions of positive order. Let us indicate by $M_p^p(n)$ the dependence of M_p^p on the dimension of the underlying space R^n. Let $a(\xi)$ be a mapping of R^n into R^m and let a^* be the induced mapping of functions on R^m into functions on R^n defined by $(a^*f)(\xi)=f(a(\xi))$. It is proved that, if a is twice continuously differentiable, and if $a^*[M_p^p(m)]\subset M_p^p(n)$, then a is necessarily affine. Conversely, if a is an affine mapping of R^n onto R^m, then a^* maps $M_p^p(m)$ isometrically into $M_p^p(n)$. This striking theorem is closely related to results of Beurling and Helson [Math. Scand. **1** (1953), 120–126; MR **15**, 307], for the case $p=1$.

Part II: Let us denote by K^a, $1\leq a<\infty$, the set of all locally integrable functions $k(x)$ on R^n for which, for some intervals I and I^* and some constant C,

$$\left[\int_{x\notin I}|k(x-y)-k(x)|^a\,dx\right]^{1/a}\leq C,\quad y\in tI^*,$$

for all $t>0$. It is proved that if $k\in K^a$, then the convolution $Tu=k*u$ is of weak type 1, a. Thus, by the Marcinkiewicz interpolation theorem, if, for some $p_0>1$, $T\in L_{p_0}^{q_0}$, where $(p_0)^{-1}-(q_0)^{-1}=1-a^{-1}$, then $T\in L_p^q$ for p and q satisfying $p^{-1}-q^{-1}=1-a^{-1}$ and $1<p\leq p_0$. Using standard duality and convexity arguments, the restriction $p<p_0$ can be removed. Various important results can be obtained by specialization. Thus, many of the results of Calderón and Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637] are included as a special case. A further application of great interest is the following improvement of a theorem of Mihlin [Vestnik Leningrad Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 143–155; MR **19**, 546]. Let m be the least integer $>n/2$ and let $f\in L^\infty(n)$ be such that for all $R>0$ and $|\alpha|\leq m$, and some constant B,

$$\int_{\frac{1}{2}R\leq|\xi|\leq 2R}|R^{|\alpha|}D_\alpha f|^2\,d\xi\leq B^2R^n.$$

Here $D_\alpha f=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}f$, $|\alpha|=\alpha_1+\cdots+\alpha_m$. Then $f\in M_p^p$.

Part III: Let C be a set in R^m such that $0\notin C$ and $C\cup\{0\}$ is a closed cone. Let $K(x,t)$ be defined and measurable on $R^n\times C$ and such that $K(ax,at)=a^{-n}K(x,t)$ for $a>0$. Consider the transformations

$$U(x,t)=\int_{R^n}K(x-y,t)u(y)\,dy,$$

$$v(x)=\int_{R^n\times C}K^\sim(x-y,t)V(y,t)t^{-m}\,dy\,dt,$$

where $K^\sim$ is the complex conjugate of $K(-x,t)$. Define on $R^n\times C$ the norm

$$X^pT^q(U)=\left\{\int_{R^n}\left[\int_C|U(x,t)|^q|t|^{-m}\,dt\right]^{p/q}dx\right\}^{1/p}.$$

A theory parallel to that of Part II is developed for the mappings $u(x)\to U(x,t)$ and $V(x,t)\to v(x)$. This theory is then specialized to extend and complete earlier work of E. M. Stein [Trans. Amer. Math. Soc. **88** (1958), 430–466; MR **22** #3778] which generalizes to n dimensions results obtained in one dimension by Marcinkiewicz, Littlewood and Paley, etc. *I. I. Hirschman, Jr.* (Erlenbach)

12390:

Freud, Géza. Sui procedimenti lineari d'approssimazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 641–643.

Soit C l'espace des fonctions $f(x)$ continues sur $[0,1]$, satisfaisant à:

$$a_if(0)+b_if(1)=0\qquad(i=1,2),$$

où les a_i et b_i sont des constantes qui peuvent être nulles toutes les quatre; soit C_2 le sous-espace des fonctions admettant une dérivée seconde continue, H une fonction linéaire et continue appliquant C sur un espace de Banach B; alors si:

$$\|Hf\|_B\leq\alpha\|f\|\quad\text{pour } f\in C,$$

$$\|Hf\|_B\leq\beta\|f''\|\quad\text{pour } f\in C_2,$$

on a, pour ν entier ≥ 1:

$$\|Hf\|_B\leq(\alpha+4\beta\nu^2)\omega_2(f;\nu^{-1})$$

où:

$$\omega_2(f;\delta)=\max_{\substack{|h|\leq\delta\\0\leq x\leq 1}}|f(x+h)-2f(x)+f(x-h)|.$$

J. Favard (Paris)

12391:

Kostyučenko, A. G.; Mityagin, B. S. Positive definite functionals on nuclear spaces. Dokl. Akad. Nauk SSSR **131** (1960), 13–16 (Russian); translated as Soviet Math. Dokl. **1**, 177–180.

Let Φ be a nuclear space with a continuous Hermitian form (ϕ,ψ). The completion of Φ to a Hilbert space H has the property that $\Phi\subset H\subset\Phi'$, where Φ' is the space adjoint to Φ. If the finite system $\{A_i\}$ of symmetric

operators have self-adjoint mutually commuting extensions in H, then this system has a complete system of generalized eigenfunctionals, $\chi_{\lambda_1 \cdots \lambda_n}$, defined on Φ, and we have the representation

$$(\phi, \psi) = \sum_n \int_{-\infty}^{\infty} \cdots \int (\chi_{\lambda_1 \cdots \lambda_n}, \phi)^* \times (\chi_{\lambda_1 \cdots \lambda_n}, \psi)\, d\sigma_n(\lambda_1, \cdots, \lambda_n),$$

where σ_n is a system of finite measures on n-dimensional space.

If Φ is a nuclear algebra with an involution $x \to x^*$, a continuous operator A will be called 'typical' if for any $x, y \in \Phi$, $Ax \circ y^* = x \circ (Ay)^*$. A continuous functional on Φ is 'positive-definite' if for all $x \in \Phi$, $T(x \circ x^*) \geqq 0$. Any positive-definite functional T defines a scalar product on Φ in a natural way, and thereby Φ can be extended to a Hilbert space H_T. Suppose $\{A_i\}$ is a finite system of typical operators which admit pairwise commutative extensions on H_T, and for each $\lambda = (\lambda_1, \cdots, \lambda_n)$ there exists no more than one generalized eigenfunctional. Then T has a representation

$$T(x) = \int_{-\infty}^{\infty} (\chi_\lambda, x)\, d\sigma(\lambda).$$

An important question is when the representing measure is unique. This happens, e.g., when the closures of the operators in $\{A_i\}$ are self-adjoint pairwise commutative operators.

In specific cases it is usually extremely difficult to decide when the system $\{A_i\}$ has commuting extensions. The authors give a partial answer which can be used to handle some special cases. Consider two nuclear algebras Φ_1 and Φ_2 with involutions, and A_1 and A_2 symmetric operators on the corresponding spaces. Suppose T is a positive-definite functional defined on the tensor product $\Phi_1 \otimes \Phi_2$ and suppose further that the closures of $I_1 \otimes A_2$ and $A_1 \otimes I_2$ are self-adjoint in H_T. Then the closures of these operators commute.

A number of examples are given of the representation of positive-definite functionals on the spaces S_a^ρ introduced by I. M. Gel'fand and G. E. Šilov [*Obobščennye funkcii i deĭstviya nad nimi*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **20** #4182].

A. Devinatz (Princeton, N.J.)

12392:

Kostyučenko, A. G.; Mityagin, B. S. The multi-dimensional problem of moments. Dokl. Akad. Nauk SSSR **131** (1960), 1249–1252 (Russian); translated as Soviet Math. Dokl. **1**, 415–419.

The authors deduce, from their general theory [see #12391], a number of results on positive-definite functionals on sequence spaces. Let $M(a_{np})$ be a countably-normed space of sequences $\phi = \{\phi_n, n = 0, 1, \cdots\}$ with the system of norms

$$|\phi|_p = \sum_{n=0}^{\infty} \phi_n a_{np}, \quad p = 1, 2, \cdots,$$

where $\{a_{np}\}$ is a countable collection of sequences of positive numbers, satisfying the conditions (a) $a_{np} \leqq a_{n,p+1}$, (b) $\sum_0^\infty a_{np}/a_{n,p+1} < \infty$, (c) $a_{n+m,p} \leqq a_{n,p+1} a_{m,p+1}$ for all n, m, p. Multiplication in M is defined by

$$(\phi\psi)_n = \sum_0^n \phi_{n-l}\psi_l, \quad n = 0, 1, \cdots.$$

Condition (b) insures we have a nuclear topology and (c) insures the continuity of multiplication. Involution is defined by complex conjugation, and the operator A is defined by $(A\phi)_n = \phi_{n-1}$, $(A\phi)_0 = 0$. Every positive functional T on M has the representation

$$T(\phi) = \int_{-\infty}^{\infty} g_\lambda(\phi)\, d\sigma(\lambda),$$

where $g_\lambda(\phi) = \sum \lambda^n \phi_n$.

A sequence s_n ($n = 0, 1, \cdots; s_n > 0$) is called 'of type K' if

$$\int_1^\infty \frac{\log S(r)}{r^2}\, dr = \infty, \quad \text{where } S(r) = \sup_{n \geqq 0} r^n/s_n.$$

It turns out that the representing measure of a positive-definite functional on M is unique if and only if all of the sequences $\{a_{np}\}$, $p = 1, 2, \cdots$, are sequences of type K. An extension of the representation and uniqueness theorem is given for k-fold sequence spaces.

If T_n, $n = 0, 1, \cdots$, is a sequence satisfying the conditions $\sum_0^n T_{n+m} \xi_n \xi_m \geqq 0$, where $\{\xi_n\}$ is any finite set of complex numbers, then it is well known that

$$T_n = \int_{-\infty}^{\infty} \lambda^n\, d\sigma(\lambda),$$

$d\sigma$ positive and bounded. It turns out that if this representing measure is unique for every positive-definite sequence T_n such that $|T_n| \leqq t_n$, $n = 0, 1, \cdots$, then $\{t_n\}$ is a sequence of type K.

In the multi-dimensional case, i.e., $n = (n_1, \cdots, n_k)$, the fact that T_n is a positive-definite sequence is, in general, not sufficient to get an integral representation [R. B. Zarhina, Dokl. Akad. Nauk SSSR **124** (1959), 743–746; MR **21** #2161]. However, additional conditions can be given which insure a unique representation, and the authors obtain results analogous to those obtained by the reviewer [Duke Math. J. **24** (1957), 481–498; MR **19**, 1047]. Several further theorems are given which are immediate consequences of the multi-dimensional representation theorem. *A. Devinatz* (Princeton, N.J.)

12393:

Ismagilov, R. S. Self-adjoint extensions of a system of commuting symmetric operators. Dokl. Akad. Nauk SSSR **133** (1960), 511–514 (Russian); translated as Soviet Math. Dokl. **1**, 867–870.

Let Φ be a linear topological space with commuting linear continuous operators A and B acting in it, and let H be a Hilbert space obtained by completing Φ with respect to a scalar product (x, y). Suppose that the closure, $\bar{A}$, of the operator A is self-adjoint in H and that B is symmetric. The author considers the question of when there exists a self-adjoint extension of B which commutes with $\bar{A}$.

The operator $\bar{A}$ is said to be strongly self-adjoint in H if Φ contains a subset Φ_1 such that (a) Φ_1 is dense in Φ in the topology of Φ; (b) if $x \in \Phi_1$, then the operator A when restricted to the linear manifold generated by the set $\{A^k x; k = 0, 1, \cdots\}$ has a self-adjoint closure in the closure of that space. Denote by Φ^λ the set $(B - \lambda E)\Phi$ and by H^λ the closure of this set in H. Theorem: Suppose that A is strongly self-adjoint in H^λ for some λ with $\operatorname{Im} \lambda \neq 0$ and there exists an involution $x \to x^*$ such that $(x, y)^* =$

(x^*, y^*) and A and B are real with respect to this involution. Then the operator B has a self-adjoint extension which commutes with $\overline{A}$.

An element $x \in \Phi$ will be called a Carleman element if there exists a sequence $\{m_k\}$ such that $\sum m_{2k}^{-1/2k} = \infty$ and the sequence $\{A^k x / m_k\}$ is bounded in the topology of Φ. An operator A will be called a Carleman operator if Φ contains a dense set of Carleman elements. From Carleman's well known criterion for the uniqueness of the moment problem it is easy to see that $\overline{A}$ is strongly self-adjoint in H^λ. It follows immediately from the above theorem that if A is a Carleman operator, and A and B are real with respect to the involution in Φ, then B admits a self-adjoint extension in H which commutes with $\overline{A}$.

Several examples of Carleman spaces are given, from which is deduced a theorem on the extension of positive-definite functions from strips.

A. Devinatz (Princeton, N.J.)

12394:

Èskin, G. I. A sufficient condition for the solvability of a multi-dimensional problem of moments. Dokl. Akad. Nauk SSSR **133** (1960), 540–543 (Russian); translated as Soviet Math. Dokl. **1**, 895–898.

Let Φ_1 and Φ_2 be linear spaces with involution, and A and B linear operators on Φ_1 and Φ_2 respectively, which are real with respect to the given involutions. Let $\Phi = \Phi_1 \otimes \Phi_2$ be the tensor product of Φ_1 and Φ_2; the operators A and B and the involution are extended to Φ in the natural way. Suppose that there is given on Φ an inner product which is real with respect to the involution, i.e., $(x^*, y^*) = (x, y)^*$, and that A and B are symmetric operators with respect to this inner product. Denote by H the completion of Φ with respect to the given inner product. For fixed $\psi_0 \in \Phi_2$ ($\phi_1 \otimes \psi_0, \phi_2 \otimes \psi_0$) is clearly an inner product in Φ_1. Designate the completion of Φ_1 with respect to this inner product by H_{ψ_0} and the closure of A in H_{ψ_0} by A_{ψ_0}.

Theorem: If A_{ψ_0} is self-adjoint for every fixed $\psi_0 \in \Phi_2$, then the closure of A in H is self-adjoint and B has a self-adjoint extension which commutes with this closure.

Applications of this theorem are given to the two-dimensional problem of moments and the two-dimensional problem of the extension of positive-definite functions. Suppose $n = (n_1, n_2)$ and $\{C_n\}$ is a sequence such that $\sum C_{n_i + n_j} \xi_i \xi_j^* \geq 0$. If for every fixed n_0 the one-dimensional moment problem $A_m = C_{m, 2n_0} + C_{m, 2(n_0+1)}$ has a unique solution, then there exists a bounded positive measure $d\sigma$ such that

$$C_n = \int \lambda^{n_1} \mu^{n_2} \, d\sigma(\lambda, \mu).$$

For the case of positive-definite functions we have the following: If $f(x, y)$ is a continuous positive-definite function on the rectangle $-a < x < a$, $-b < y < b$, and if the positive-definite function $f(x, 0)$ has a unique positive-definite extension, then $f(x, y)$ has a positive-definite extension to the whole plane. These results are extensions of theorems obtained by the reviewer [Duke Math. J. **24** (1957), 481–498; Acta Math. **102** (1959), 109–134; MR **19**, 1047; **22** #875]. *A. Devinatz* (Princeton, N.J.)

12395:

Putnam, C. R. On differences of unitarily equivalent self-adjoint operators. Proc. Glasgow Math. Assoc. **4**, 103–107 (1960).

Let A be a bounded, selfadjoint operator and U a unitary operator in a Hilbert space. Several necessary conditions are deduced for $H = A - U^* A U \geq 0$. For example: $\|H\| \leq \delta(A)$ ($\delta(A)$ is the difference of the maximum and the minimum points of the essential spectrum of A); if $H \neq 0$, the spectrum of U has a positive linear measure; the point spectrum of U has no more than N points, multiplicities counted (N is the multiplicity of the eigenvalue 0 of H); if $N < \infty$, the continuous spectrum of U is absolutely continuous; if $N = 0$, the maximum and minimum points of the spectrum of A cannot belong to the point spectrum. Applications of these results to semi-normal operators D ($DD^* - D^*D$ is semi-definite) are considered. *T. Kato* (Tokyo)

12396:

Šmul'yan, Yu. L. An operator Hellinger integral. Mat. Sb. (N.S.) **49 (91)** (1959), 381–430. (Russian)

Soient X un ensemble, R un anneau de parties de X, H un espace hilbertien. Soit Φ une fonction (non nécessairement additive) définie sur R, dont les valeurs sont des opérateurs de H dans un espace hilbertien H', telle que $\Phi(\emptyset) = 0$. On dit que Φ est [faiblement, fortement, uniformément] intégrable sur $M_0 \in R$ si les sommes $\sum_k \Phi(M_k)$ ont une limite [faible, forte, uniforme] suivant l'ensemble filtrant des partitions finies (M_k) de M_0. La limite est notée $\int_{M_0} \Phi(dM)$ et est appelée l'intégrale de Φ sur M_0. L'intégrale est héréditaire et additive. Si les valeurs de Φ sont des opérateurs hermitiens de H, si Φ est sous-additive et si l'ensemble des sommes $\sum_k \Phi(M_k)$ est borné, alors Φ est fortement intégrable sur M_0 et $\int_{M_0} \Phi(dM) = \sup \sum_k \Phi(M_k)$. Si les valeurs de Φ sont des opérateurs positifs de H et si Φ est sous-additive, alors l'intégrale est monotone; si, en outre, Φ est dénombrablement sous-additive, alors l'intégrale est fortement dénombrablement additive. Si H est l'espace des nombres réels, on retrouve l'intégrale de Kolmogorov [Math. Ann. **103** (1930), 654–696].

Supposons que $H = H_1 \oplus H_2$ et soit le système

$$\mathfrak{A} = \begin{pmatrix} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

des opérateurs A_{ik}: $H_k \to H_i$ $(i, k = 1, 2, i + k > 2)$, tel que $A_{22} \geq 0$, $A_{21} = A_{12}^*$ et $\sup_{g \in H_2} \|A_{12}\|^2 / (A_{22} g, g) < +\infty$. Alors il existe un opérateur $A_{11} = A_{11}^*$ dans H_1 tel que l'opérateur A défini dans H par $Af = A_{1k} f + A_{2k} f$ pour $f \in H_k$ $(k = 1, 2)$ soit positif. Parmi les opérateurs A_{11} il existe un plus petit noté $\mathfrak{A}^\wedge$ et on a

$$(\mathfrak{A}^\wedge f, f) = \sup_{g \in H_2} |(f, A_{12} g)|^2 / (A_{22} g, g)$$

pour $f \in H_1$. On a $\mathfrak{A}^\wedge \geq 0$. Si A_{22} est inversible, alors $\mathfrak{A}^\wedge = A_{12} A_{22}^{-1} A_{21}$. {Pour l'essentiel, ces propositions ont été trouvées antérieurement par C. Davis [Trans. Amer. Math. Soc. **87** (1958), 144–158; MR **19**, 1173].}

Supposons maintenant que A_{12}, A_{22}, A_{21} sont des fonctions dénombrablement additives définies sur R. Alors $\mathfrak{A}^\wedge$ est une fonction dénombrablement sous-additive définie sur R, dont les valeurs sont des opérateurs positifs de H_1. On dit que le système correspondant $\mathfrak{A}$ est intégrable au sens de Hellinger sur $M_0 \in R$ si la fonction $\mathfrak{A}^\wedge$ est intégrable sur M_0. L'intégrale $\int_{M_0} \mathfrak{A}^\wedge(dM)$ est appelée l'intégrale de Hellinger de la fonction $\mathfrak{A}$ et est notée

$\int_{M_\sigma} A_{12}(dM)A_{22}^{-1}(dM)A_{21}(dM)$. L'auteur établit les propriétés de cette intégrale et donne quelques applications concernant les fonctions matrices (à valeurs opérateurs) harmoniques sur le cercle unité, et les fonctions spectrales de certains opérateurs autoadjoints non nécessairement bornés.

Remarque: L'auteur annonce qu'après la dernière ligne de la page 402 il faut ajouter: $\sup_{p\in P}(A(p)f,f)\equiv a(f)<+\infty$ (3.1). *N. Dinculeanu* (Bucharest)

12397:

Dean, David; Raimi, Ralph A. Permutations with comparable sets of invariant means. Duke Math. J. **27** (1960), 467–479.

Let X be an infinite set. Let σ be a motion of X, i.e., a one-to-one mapping of X into itself. Let S (the representation of σ) be the linear transformation of the space E of all bounded real-valued functions on X into itself such that $Sf(p)=f(\sigma p)$, for $p\in X$ and $f\in E$. A positive linear functional ϕ' on E with $\|\phi'\|=1$ and such that $(\phi',f)=(\phi',Sf)$, for any f in E, is said to be a σ-invariant mean. Let M_σ' denote the set of all σ-invariant means.

"The object of the present paper is to provide several answers to the following question: Given two motions σ and μ, with representations S and U, what properties of σ and μ, or of S and U, characterize the situation $M_\sigma'\subset M_\mu'$?" A typical equivalent condition for $M_\sigma'\subset M_\mu'$ which is obtained is $\lim_n\|S_nU-S_n\|=0$, where $S_n=n^{-1}\sum_{i=1}^n S^i$. The last part of the paper is devoted to some investigation of motions of the set N of all integers. It is then shown how every motion without finite orbits is equivalent to a motion having a particularly simple appearance. Here the equivalence of the motions σ and μ means that $M_\sigma'=M_\mu'$. E.g., it is proved that, if σ is a motion of N, then there exists a motion μ equivalent to σ which is an onto motion.

{Editor's note: The authors communicate the following errata. In Theorems 3.3 and 4.2 it is necessary to add the hypothesis that $F_\sigma=F_\mu$, where F_σ is the collection of finite cycles in σ, as in the definition preceding Lemma 4.3. This error does not affect what follows Theorem 4.2 and is irrelevant to what precedes Theorem 3.3. On p. 468, l. 6: the first $\bigcup$ should be $\bigcap$; p. 468, l. 19: $S_\alpha p_\alpha$ should be $S_\alpha' p_\alpha'$; p. 472, l. 27: S_n should be S_{n_i}; p. 473, l. 22: "$\|f\|1$" should be "$\|f\|=1$".} *A. Pelczyński* (Warsaw)

12398:

Solomyak, M. Z. Analyticity of a semigroup generated by an elliptic operator in L_p spaces. Dokl. Akad. Nauk SSSR **127** (1959), 37–39. (Russian)

If $-A$ is the infinitesimal generator of a bounded semigroup of operators $T(\xi)$, then the fractional powers A^α, $0<\alpha<1$, can be defined. A second operator B is said to be of fractional order $\alpha<1$ with respect to A if $D(B)\supset D(A)$ and if for each $r>\alpha$ with $r<1$ there exists a constant K_r such that $|Bx|\le K_r|A^rx|$ for all $x\in D(A)$. In order that B be of fractional order α with respect to A when $T(\xi)$ is analytic, it is necessary that for each $x\in D(A)$, $r>\alpha$, $\delta>0$, the inequality $|Bx|\le K_1\delta^{1-r}|Ax|+K_2\delta^{-r}|x|$ hold, where K_1 and K_2 do not depend on δ or x. This result is then applied to elliptic operators

$$Au=(-1)^m\sum_{|\kappa|=2m}a_\kappa(x)\partial^{2m}u/\partial x^\kappa+Tu$$

defined over a bounded region Ω in R_n with sufficiently smooth boundary Ω and boundary conditions $\partial^k u/\partial n^k=0$, $0\le k\le m-1$, the coefficients a_κ being real and m-times continuously differentiable and satisfying the condition of uniform ellipticity. Theorem 1: If the operator A is self-adjoint and positive definite, then for $f\in L_p(\Omega)$, $p>1$, and each $\lambda\le 0$, the equation $\lambda u-Au=f$ has a unique solution $u\in W_p^{2m}(\Omega)$ and $|u|_{W_p^{2m}}\le c_p|f|_{L_p}$. This inequality holds uniformly in λ in any sector S_φ: $\varphi<\arg\lambda<2\pi-\varphi$. Theorem 2: If A is merely strongly elliptic of order $2m$ and is the sum of a self-adjoint positive definite operator and a skew-symmetric operator, then there is a $\varphi_0<\frac{1}{2}\pi$ such that for each $\lambda\in S_\varphi$ ($\varphi>\varphi_0$) the above problem has a unique generalized solution $u\in W_p^{2m}(\Omega)$ satisfying the above inequality uniformly with respect to $\lambda\in S_\varphi$. Theorem 3: If A fulfills the conditions of either Theorems 1 or 2 above, then for $\lambda\in S_\varphi$, $|R(\lambda;A)|_{L_p}\le c(|\lambda|+1)^{-1}$ and A is the generator of a semi-group of operators analytic in some sector. Theorem 4: For each $k<2m$ each differentiation operator of order k is of fractional order $k/2m$ with respect to A. Finally, if B is any differential operator of order $<2m$ with bounded coefficients, then $A+B$ satisfies the assertions of Theorems 1–3 for each $\lambda\in S_\varphi$ outside of a sufficiently large circle. *R. S. Phillips* (Stanford, Calif.)

12399:

Yosida, Kôsaku. Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them. Proc. Japan Acad. **36** (1960), 86–89.

Let $[T(t);t\ge 0]$ be a strongly continuous bounded semi-group of operators on a Banach space with infinitesimal generator A. A fractional power $-(-A)^\alpha$, $0<\alpha<1$, of A was defined by S. Bochner [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 368–370; MR **10**, 720] and the reviewer [Pacific J. Math. **2** (1952), 343–369; MR **14**, 383] as the infinitesimal generator of the semi-group $T_\alpha(t)x=\int_0^\infty T(s)x\,d\gamma_{t,\alpha}(s)$, where the measure $d\gamma_{t,\alpha}(s)\ge 0$ is defined through the Laplace integral $e^{-ta^\alpha}=\int_0^\infty e^{-\lambda s}\,d\gamma_{t,\alpha}(\lambda)$. The author finds an explicit representation for the measure $d\gamma_{t,\alpha}(\lambda)$, and using this representation, he shows that the semi-group $T_\alpha(t)$ has an analytic extension into a sector of the complex t-plane containing the positive real axis.

R. S. Phillips (Stanford, Calif.)

12400:

Kato, Tosio. Note on fractional powers of linear operators. Proc. Japan Acad. **36** (1960), 94–96.

Extending the results of the preceding paper by K. Yosida [#12399], the author proves for a large class of linear operators A that the fractional power A^α ($0<\alpha<1$) generates a semi-group of operators $T_\alpha(t)$ which has an analytic extension in a sector of the complex t-plane containing the positive real axis. More precisely, he treats operators A of class (ω, M) with the following properties: (i) A is densely defined and closed; (ii) the resolvent set of $-A$ contains the open sector $|\arg\lambda|<\pi-\omega$, $0<\omega<\pi$, and $\lambda(\lambda+A)^{-1}$ is uniformly bounded in norm in each smaller sector $|\arg\lambda|<\pi-\omega-\varepsilon$, $\varepsilon>0$; in particular, $\lambda\|(\lambda+A)^{-1}\|\le M$, $\lambda>0$. If A is of type (ω, M), then the fractional power A^α, $0<\alpha<1$, can be defined through

$$(\lambda+A^\alpha)^{-1}=$$

$$(\pi)^{-1}\sin\pi\alpha\int_0^\infty \mu^\alpha[\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}]^{-1}(\mu+A)^{-1}\,d\mu$$

which is valid for λ on and near the positive real axis. The operator A^α is of type $(\alpha\omega, M)$, and for $\alpha\omega < \pi/2$, $-A^\alpha$ is the infinitesimal generator of an analytic semi-group of operators. *R. S. Phillips* (Stanford, Calif.)

12401:

Langlands, R. P. On Lie semi-groups. Canad. J. Math. **12** (1960), 686–693.

Let a semi-group structure be defined on the first quadrant $\pi = \{(p_1, \cdots, p_n);\ p_1 \geq 0, \cdots, p_n \geq 0\}$ of E_n by $(p, q) \to F(p, q) = p \circ q$. Consider a representation $T(\cdot)$ of π as a semi-group of bounded linear operators on a Banach space X. If we assume the postulates P_1–P_6 of Chapter 25 of E. Hille and R. S. Phillips' book [*Functional analysis and semi-groups*, Amer. Math. Soc., Providence, R.I., 1957; MR **19**, 664], then there is a continuous functional $f(\cdot)$ defined on π such that $f((\rho+\sigma)a) = f(\rho a) \circ f(\sigma a)$ for $a \in \pi$, $\rho, \sigma \geq 0$, so that the representation $\rho \to T(f(\rho a))$ gives rise to a one-parameter semi-group, strongly continuous and with $T(0) = I$. Let $A(a)$ be its infinitesimal generator. The author studies the relations among the $A(a)$ and their adjoints $A^*(a)$, assuming, as in Hille-Phillips [loc. cit.], that $F(p, q)$ is three times continuously differentiable. He first proves some "dense graph theorems" which are suggested in E. Hille, Bull. Amer. Math. Soc. **56** (1950), 89–114 [MR **12**, 10]. Using these he shows that the expected linear and commutation relations hold for $A(a)$, and also for $A^*(a)$. *K. Yosida* (Tokyo)

12402:

Civin, Paul. Isometries of group algebras. Proc. Amer. Math. Soc. **11** (1960), 983–985.

Let G be a locally compact abelian group, $\hat{G}$ its character group, and A the group algebra of G. Associated with any automorphism φ of A is a homeomorphism τ of $\hat{G}$ onto itself. Helson [Ark. Mat. **2** (1953), 475–487; MR **15**, 327] and Wendel [Pacific J. Math. **1** (1951), 305–311; MR **14**, 246] have shown the equivalence of the following two statements: (a) φ is an isometry, and (b) $\tau(e)\tau(xy) = \tau(x)\tau(y)$, for all $x, y \in \hat{G}$, where e is the unit of $\hat{G}$. The author establishes a third equivalent condition: (c) $\varphi T_\alpha = T_{\rho(\alpha)}\varphi$, where φ is a homomorphism of A onto A, T_α, $\alpha \in \hat{G}$, is the operator on A defined by $(T_\alpha f)(x) = f(x)(x, \alpha)$ $(f \in A, x \in G)$, and ρ is a map of $\hat{G}$ into $\hat{G}$.
Ti Yen (E. Lansing, Mich.)

12403:

Katznelson, Y. Sur les algèbres dont les éléments non négatifs admettent des racines carrées. Ann. Sci. École Norm. Sup. (3) **77** (1960), 167–174.

Let B be a commutative semi-simple self-adjoint Banach algebra. The author proves that if every positive $f \in B$ has a positive square root, then B is a $C(X)$. (This was a long-standing and very natural conjecture. For the mapping $f \to \sqrt{f}$, defined on the whole positive cone, ought to be continuous, hence $|f|^2 \leq \text{const}\,|f^2|$, hence $B = C(X)$ by the C^* axioms.)

The proof in the present paper was later improved [#12404]. *H. Mirkil* (Hanover, N.H.)

12404:

Katznelson, Yitzhak. A characterization of the algebra of all continuous functions on a compact Hausdorff space. Bull. Amer. Math. Soc. **66** (1960), 313–315.

Let B be a commutative semisimple self-adjoint regular Banach algebra with 1, and let M be its maximal ideal space. Theorem: Suppose that in each (semisimple) quotient algebra of B the idempotents are bounded (the bound perhaps depending on the quotient algebra). Then $B = C(M)$. Corollary: Let F be defined on an interval $(-\delta, \delta)$ with $F(0) = 0$, and $x^{-1}F(x) \to +\infty$ as $x \to 0$. Suppose that for each $g \in B$ with spectrum in $(-\delta, \delta)$ we have the composite function $F(g) \in B$. Then $B = C(M)$.

{Reviewer's remarks: (1) The main theorem of the author's previous paper [#12403], that $B = C(M)$ if every positive $g \in B$ has a square root, follows immediately. For define $F(x) = x|x|^{-1/2}$, $F(0) = 0$. (This F operates, because $|x|$ operates.) (2) The condition $x^{-1}F(x) \to +\infty$ is also necessary. For suppose $x_k \to 0$ with $x_k^{-1}F(x_k)$ bounded. Then take B to be, say, all continuous g on $[-1, 1]$ with $\sup |g(x_k)/F(x_k)| = \|g\| < \infty$. (3) The author's method cannot handle an F with one-sided domain $[0, \delta)$.}
H. Mirkil (Hanover, N.H.)

12405:

Wermer, John. Dirichlet algebras. Duke Math. J. **27** (1960), 373–381.

A closed subalgebra A of $\mathscr{C}(X, \mathbf{C})$ where X is compact, is called a 'Dirichlet algebra' if $1 \in A$, A separates points of X, and the real parts of the functions in A are everywhere dense in $\mathscr{C}(X, \mathbf{R})$. Let $\mathfrak{M}$ be the subset of the dual A' of multiplicative, non-zero, functionals. Following A. M. Gleason, one defines $x \sim y$ for $x, y \in \mathfrak{M}$ if $\|x - y\| < 2$, using the usual norm in A'. Gleason has shown that this is an equivalence relation, and has called the corresponding classes 'parts' of $\mathfrak{M}$. The main theorem in this paper is that (for Dirichlet algebras) each part P either is a point or is homeomorphic ($h: \{|z| < 1\} \to P$) with a disc such that $f \circ h$ is holomorphic for each f in A. The proof involves harmonic analysis in the L^2 of the measure on X representing a point m_0 of $\mathfrak{M}$ which lies in a non-trivial part. Let H^2 be the closed subspace of L_2 generated by A itself, and suppose C is an A-submodule of H^2. Then there exists an $E_0 \in C$, $|E_0| = 1$ a.e., which is a factor of every $F \in C$. Three pages of intricate steps show that E_0^{-1} provides the desired mapping. There is a list of unsolved problems. *R. Arens* (Los Angeles, Calif.)

12406:

Harris, R. T. Generalized eigenfunction expansions for operator algebras. Trans. Amer. Math. Soc. **98** (1961), 485–500.

Let A denote a uniformly closed abelian *-algebra of operators on a Hilbert space, H, and V the spectrum of A. There is a well known mapping of the complex-valued continuous functions f on V into A; if f is such a function, let T_f denote the corresponding transformation of A. If x and x' are elements of H, then the inner product $(T_f x, x')$ has the character of an integral over V, whose measure is denoted $m_{x,x'}$. A measure v on V which bounds the $m_{x,x'}$ permits one to construct Hilbert spaces of square-summable functions on V, and these in turn are mapped onto H by extending the mapping $f \to T_f x$ for a given x in H. When certain dense subsets, D, of H are prescribed, one can obtain representation of elements of

H as integrals over V, and such representations are considered "generalized eigenfunction expansions". The author considers the relations of D and H which permit such representations, in particular in terms of the mappings obtained by closing the finite dyadic expressions of D' and H in the greatest cross-norm of Schatten. (D is considered to be a Banach space and D' is the set of linear functionals on D.) Conditions are obtained which imply that the set D will permit the "generalized eigenfunction expansions" for any A. The author's formulation of the problem is quite abstract, but the historical references given should be adequate to permit a correspondence with the more direct notions of eigenfunction expansions. *F. J. Murray* (New York)

12407:

Goldberg, Seymour. Some properties of the space of compact operators on a Hilbert space. Math. Ann. **138** (1959), 329–331.

Results: $\mathscr{K}$, the set of compact operators on a separable Hilbert space $\mathscr{H}$, is not isomorphic to any conjugate space X' of a Banach space X [cf. R. Schatten, Math. Ann. **134** (1957), 47–49; MR **19**, 756; where "isomorphic" above is replaced by "equivalent"]. Corollaries, depending on the results of C. Bessaga and A. Pelczynski [Studia Math. **17** (1956), 151–164; Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 249–250; MR **22** #5872, 5873], discuss the existence of projections of $\mathscr{B}$, the set of bounded operators on $\mathscr{H}$, onto various subspaces. In particular, there is no projection of $\mathscr{B}$ onto $\mathscr{K}$.

B. Gelbaum (Minneapolis, Minn.)

12408:

Dixmier, Jacques. Sur les C^*-algèbres. Bull. Soc. Math. France **88** (1960), 95–112.

If A is a C^*-algebra, $A^\wedge$ will denote the space of all primitive ideals of A, equipped with the Jacobson hull-kernel topology. A is CCR if T_a is completely continuous for all a in A and all irreducible *-representations T of A. For CCR algebras A, Kaplansky [Trans. Amer. Math. Soc. **70** (1951), 219–255; MR **13**, 48] has proved the following three properties of $A^\wedge$: (I) $A^\wedge$ is a Baire space (i.e., a countable union of closed nowhere dense sets has no interior points); (II) if A is not primitive, one can find two disjoint non-void open subsets of $A^\wedge$ (that is, if A has no ideal divisors of 0, it is primitive); (III) $A^\wedge$ has a dense open subset which is Hausdorff in the relativized topology. To what extent are these properties valid for general C^*-algebras? Using recent results, especially Kadison's irreducibility theorem [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276; MR **19**, 47], the author shows the following: Property (I) holds for all C^*-algebras. Property (II) holds for all separable C^*-algebras (whether it holds in general is still unknown). As for (III), the author constructs a C^*-algebra A such that $A^\wedge$ has no non-void open Hausdorff subset.

A GCR algebra is a C^*-algebra having a (transfinite) composition series all of whose quotients are CCR. It is known that a GCR algebra A has the following property: (IV) If T and T' are two irreducible *-representations of A having the same kernel, then T and T' are equivalent. In the last part of this article, the author proves the converse for the separable case: If a separable C^*-algebra satisfies (IV), it is GCR. In the process he obtains the following interesting generalization of a theorem of Rosenberg [Amer. J. Math. **75** (1953), 523–530; MR **15**, 236]: Let H be a Hilbert space and A a separable irreducible C^*-algebra of operators in H; and suppose further that every faithful irreducible representation of A is equivalent to the identity representation. Then A contains the completely continuous operators on H.

J. M. G. Fell (Cambridge, Mass.)

12409:

Kadison, Richard V.; Singer, I. M. Triangular operator algebras. Fundamentals and hyperreducible theory. Amer. J. Math. **82** (1960), 227–259.

Without assuming any topological closedness, an algebra $\mathscr{T}$ of bounded operators acting on a Hilbert space $\mathscr{H}$ is called a triangular algebra with a diagonal $\mathscr{A}$ provided that $\mathscr{A}=\mathscr{T}\cap\mathscr{T}^*$ is a (self-adjoint) maximal abelian algebra. The definition is also applicable to subalgebras in a factor on $\mathscr{H}$. A triangular algebra is, by Zorn's Lemma, contained in a maximal triangular algebra having the same diagonal. A projection invariant under $\mathscr{T}$ is called a hull of $\mathscr{T}$. It is proved that the hulls of a maximal triangular algebra are totally-ordered. The core of $\mathscr{T}$ is the von Neumann algebra generated by the hulls of $\mathscr{T}$. The core of a triangular algebra is a von Neumann subalgebra of the diagonal, whence the core is abelian. If the core is reduced to the scalars, the triangular algebra is called irreducible. An example of an irreducible triangular algebra is the algebra (algebraically) generated by an infinite-dimensional maximal abelian algebra $\mathscr{A}$ and a unitary operator U which acts ergodically on $\mathscr{A}$.

After these preliminary considerations, the authors are concerned with the hyperreducible case. A triangular algebra is hyperreducible provided that the core coincides with the diagonal. A maximal triangular hyperreducible algebra is frequently called by the authors an "ordered basis". An ordered basis is called discrete [continuous] when the diagonal is totally atomic [non-atomic]. A general principle of the construction of ordered bases is this: If $\{E_\alpha\}$ is a totally-ordered family of projections generating a maximal abelian algebra, then the set of all operators which leave each E_α invariant is an ordered basis. Each ordered basis arises in this way. For discrete or continuous ordered bases, the authors give more precise information. A discrete ordered basis is determined within unitary equivalence by the order type of the hulls. A continuous ordered basis acting on a separable space is unitarily equivalent to the algebra of all operators on $L_2[0, 1]$ leaving each F_λ invariant, where F_λ is the projection due to multiplication by the characteristic function of $[0, \lambda]$. Unlike the theory of abelian von Neumann algebras, a general (mixed) ordered basis cannot be decomposed into the direct sum of continuous and discrete parts, since the order, which is a proper constituent of the basis, places the atoms throughout the continuous portion of the diagonal in a manner which does not permit a separation. Under these circumstances, the authors introduce the notion of the hull class of an ordered basis acting on a separable space, which seems to have an affinity with the weighted spectrum in Wecken's theory of unitary invariants. The hull class $h(\mathscr{T})$ of a separable ordered basis $\mathscr{T}$ with the hulls $\{E_\alpha\}$ is the class of $\{(E_\alpha x, x)\}$ where x ranges over all separating unit vectors. The hull class determines an ordered basis within unitary equivalence. The authors' painstaking work on hull

classes gives further details, with order- and measure-theoretic techniques.

No application is presented in this paper. However, the authors expect that the theory is applicable to the reducibility of operators and to the analysis of infinite-dimensional representations of solvable Lie groups.

M. Nakamura (Osaka)

12410:

Civin, Paul. Involutions on locally compact rings. Pacific J. Math. **10** (1960), 1199–1202.

An involution * on a ring R is a mapping $x \to x^*$ that is additive, of period two and such that $(xy)^* = y^*x^*$. If furthermore $xx^* = 0$ if and only if $x = 0$, * is called 'proper'. The author first determines a canonical form for a proper involution * on R, a total matrix ring over a division ring D. It then follows that if D is topological and such that any involution is continuous, every proper involution of R is continuous. By using structure theorems of Kaplansky [Amer. J. Math. **69** (1947), 153–183; **70** (1948), 447–459; MR **8**, 434; **9**, 562], the author can then derive: If R is a semisimple locally compact ring (compact or connected) then any proper involution * on R is continuous.

B. R. Gelbaum (Princeton, N.J.)

12411:

Segal, I. E. A note on the concept of entropy. J. Math. Mech. **9** (1960), 623–629.

Let A be a ring of operators on complex hilbert space H and let a regular state E of A be a state which is continuous in the strong operator topology. The entropy of E with respect to a distinguished gage m on A is $-m(D \log D)$ if the last exists as an extended real number and $D = dE/dm$. If D is a bounded operator, then E is a bounded state. Theorem 1: The entropy is a concave function on bounded states. Theorem 2: The entropy of the restriction of a bounded state to a subring is not less than the entropy of the original state. States of maximal and minimal entropy are given for particular rings. Some open questions are discussed.

S. Sherman (Detroit, Mich.)

12412:

Gel'man, A. E. Theorems on an implicit abstract analytic function. Dokl. Akad. Nauk SSSR **132** (1960), 501–503 (Russian); translated as Soviet Math. Dokl. **1**, 578–580.

This paper is concerned with implicit-function theorems in complex Banach spaces. The basic context is the theory of analytic functions and their F-power series expansions, as presented by E. Hille [*Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, New York, 1948; MR **9**, 594]. The first theorem is concerned with the solution $y = y(\phi)$ of the equation $y = \Omega(y, \phi)$, where Ω is analytic in y and ϕ near (0, 0), when Ω and its first partial Fréchet derivative with respect to y vanish at (0, 0). Proofs are not given, but it is explained that the technique is that of majorant series. This first theorem, both in results and technique, is in the main familiar to workers in this general theory, as far back as Hildebrandt and Graves in 1927 and A. D. Michal and his school at the California Institute of Technology in the 1930's. The third theorem is an application of the first one. The second theorem contains results which appear to the reviewer to constitute the main contribution. Let Y, Φ, Z be complex Banach spaces. Let $\tilde{Y}$ be a finitely open set in Y, containing 0; let $\tilde{\Phi}$ be a neighborhood of 0 in Φ. Suppose $\Omega : \tilde{Y} \times \tilde{\Phi} \to Z$, with $\Omega(0, 0) = 0$; suppose Ω is G-differentiable at (0, 0) and suppose the partial G-derivative with respect to y of Ω at (0, 0) is a possibly unbounded linear operator with bounded inverse I^{-1} defined on all of Z. Finally, suppose that $\Omega(I^{-1}z, \phi)$ is analytic in z and ϕ near (0, 0). Then there exists an analytic function $y(\phi)$ (defined near 0 in Φ) such that $\Omega[y(\phi), \phi] = 0$, and having certain further properties.

Applications to classical analysis, in particular to the theory of differential equations, are mentioned.

A. E. Taylor (Los Angeles, Calif.)

12413:

Michael, J. H. Completely continuous movements in topological vector spaces. Proc. Glasgow Math. Assoc. **3**, 135–141 (1957).

Let X be a convex linear topological space containing a closed subset A. Let $1 - f$ be a completely continuous map of $A \to X$, where 1 is the identity map, for which $f\dot{A} \cap f\underline{A} = \emptyset$, where $\dot{A}$ is the boundary and $\underline{A}$ the interior of A. The author proves that then $f\dot{A}$ is the boundary of fA. The demonstration is in terms of the degree of a map.

D. G. Bourgin (Urbana, Ill.)

12414:

Jurkat, W. B. Polynomial approximation of continuous functions on linear spaces. Math. Z. **74** (1960), 99–104.

Let X and Y be Fréchet spaces over the same (real or complex) field, let D be a separable closed subset of X and R a subspace of Y. For any set G of continuous linear functionals on X, define P_G to be the set of all polynomials p in functionals g and their complex conjugates $\bar{g}$, $g \in G$, with coefficients in R. That is, P_G consists of all functions from D into R of the form

$$p(x) = \sum y_{n_1,\cdots,n_k} g_1(x)^{n_1} \cdots g_k(x)^{n_k}$$

where $x \in D$, g_i or $\bar{g}_i \in G$, $y_{n_1,\cdots,n_k} \in R$ and the sum is finite. Under the additional assumption that D satisfies a certain boundedness condition (which is fulfilled if D is bounded, or if X is a Banach space), the following theorem is proved: There exists a countable set G of functionals such that every continuous function from D into R is the limit of a sequence of polynomials from P_G (under uniform convergence on the compact subsets of D). Simple examples show that the assumptions that D be closed and separable are needed. If X and Y are Banach spaces, if $G = X^*$, and if f is a continuous function from X to Y, then a necessary and sufficient condition that f be the limit of a sequence from P_G (under uniform convergence on bounded subsets of X) is that f be weakly uniformly continuous on each bounded subset of X.

R. R. Phelps (Berkeley, Calif.)

12415:

Melamed, V. B. Calculation of the index of a fixed point of a completely continuous vector field. Dokl. Akad. Nauk SSSR **126** (1959), 501–504. (Russian)

The author announces several results concerning completely continuous operators having the form $A\varphi = B\varphi + C_n\varphi + C_{n+1}\varphi + \cdots + C_{n+k}\varphi + D\varphi$, where B is the

Fréchet derivative of A and the operators C_i and D satisfy the conditions

$$C_i(\alpha\varphi) = \alpha^i C_i\varphi,$$

$$(1)\quad \|C_i\varphi - C_i\psi\| \leq M \max\left(\|\varphi\|^{i-1}, \|\psi\|^{i-1}\right)\|\varphi-\psi\|,$$

$$\lim_{\|\varphi\|\to 0} \|D\varphi\|\,\|\varphi\|^{-n-k} = 0.$$

Let E be the Banach space with operator A acting in it, and E_1 the subspace of E formed by the eigenvectors of B corresponding to the eigenvalue 1. Let P be the projection of E on E_1. It is also assumed that $A0=0$, $PC_n x = \cdots = PC_{n+k-1}x = 0 \quad (x \in E_1)$, $PC_{n+k}x \neq 0 \quad (x \neq 0, x \in E_1)$, $k < n-1$.

Then 0 is an isolated fixed point for A and its index γ is $(-1)^\beta \gamma_1$, where β is the sum of the multiplicity orders of the eigenvalues of B greater than 1, and γ_1 represents the rotation of the field $PC_{n+k}x$ on the unit sphere in E_1. When $\gamma_1 \neq 1$, then the number 1 is a bifurcation point for A and the eigenvectors of a continuous branch satisfy the condition $\|\varphi_\mu\| = O(|\mu-1|^{1/(n+k-1)})$.

Analogous results are given for a great sphere, the conditions (1) being replaced by

$$C_i(\alpha\varphi) = \alpha^{r_i} C_i\varphi, \quad \|C_i\varphi - C_i\psi\| \leq M\|\varphi-\psi\|^{r_i},$$

$$\lim_{\|\varphi\|\to\infty} \|D\varphi\|\,\|\varphi\|^{-r_{k+1}} = 0,$$

$$1 > r_1 > r_2 > \cdots > r_k > r_{k+1} > 0, \quad r_1^2 < r_k.$$

G. Marinescu (Bucharest)

12416:

Granas, A. Homotopy extension theorem in Banach spaces and some of its applications to the theory of nonlinear equations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 387–394. (Russian. English summary)

Let E $(=E_\infty)$ denote an infinite-dimensional Banach space, let $P = E \setminus \{0\}$, and let $X \subset E$. A continuous map $f(x) = x - F(x)$ of X into E with $F(X)$ relatively compact is called a completely continuous vector field on X; the set of all such vector fields is denoted by $\mathfrak{C}(E^X)$ and the set of all non-vanishing ones by $\mathfrak{C}(P^X)$. Two elements $f(x) = x - F(x)$, $g(x) = x - G(x)$ in $\mathfrak{C}(P^X)$ are said to be homotopic in $\mathfrak{C}(P^X)$ if there is a continuous map H of $X \times I$, where $I = [0, 1]$, into E with relatively compact range such that $H(x, 0) = F(x)$, $H(x, 1) = G(x)$ and $h(x, t) = x - H(x, t)$ in $\mathfrak{C}(P^X)$. The basic tool of this paper is the following generalization of a homotopy extension theorem proved by K. Borsuk [Fund. Math. **28** (1937), 99–110] for finite-dimensional spaces. (I) Let X_0 be closed in X and let f_0, g_0 be maps of X_0 into E which are homotopic in $\mathfrak{C}(P^{X_0})$. Suppose f in $\mathfrak{C}(P^X)$ is an extension of f_0. Then there exists g in $\mathfrak{C}(P^X)$ which is an extension of g_0 such that f, g are homotopic in $\mathfrak{C}(P^X)$. The proof of this theorem is brief and elementary. If X_1 is a bounded domain in E and X_0 is the boundary of X_1, then a map f_0 in $\mathfrak{C}(P^{X_0})$ is inessential if there is an extension f in $\mathfrak{C}(P^X)$ where $X = X_0 \cup X_1$, and is essential otherwise. An immediate consequence of (I) is the fact that the property of being essential is invariant under homotopy. Using his homotopy extension theorem, the author establishes: (II) a generalization of a theorem of K. Borsuk [Fund. Math. **20** (1933), 177–190] on the essential character of antipodal maps in $\mathfrak{C}(P^S)$, $S = \{\|x\| = 1\}$; (III) the fixed point theorems due to J. Schauder, E. Rothe, and M. A. Krasnosel'skiĭ; (IV) the invariance of domain theorem of J. Schauder; (V) a result of J. Leray and J. Schauder on the continuation of solutions of equations; and (VI) a generalization of theorems of Borsuk [Fund. Math. **21** (1933), 236–243] on functions which are "ε-mappings" and "ε-mappings in the narrow sense". This paper makes no use of homology theory or of topological degree. Not all of the details are given, but the note is reasonably complete. *R. G. Bartle* (Urbana, Ill.)

12417:

Granas, A. On the disconnection of Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 395–399. (Russian. English summary)

We employ the notation introduced in the preceding review. The main result announced here is that if X is a closed bounded subset of E, then two points x_1, x_2 in $E \setminus X$ are in the same component of $E \setminus X$ if and only if the functions defined by $f_i(x) = x - x_i$, $i = 1, 2$, are homotopic in $\mathfrak{C}(P^X)$. Hence $E \setminus X$ is connected if and only if any two functions in $\mathfrak{C}(P^X)$ are homotopic in $\mathfrak{C}(P^X)$. This extends a result of K. Borsuk [Math. Ann. **106** (1932), 239–248]. The details of the present paper are presented in Fund. Math. **48** (1959/60), 189–200 [MR **22** #3967].

R. G. Bartle (Urbana, Ill.)

12418:

Gęba, K.; Granas, A.; Jankowski, A. Some theorems on the sweeping in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 539–544. (Russian summary, unbound insert)

Let X be a closed bounded subset of the Banach space E. Let $f: X \times I$ into E be a completely continuous deformation of X into a closed set Y in the sense that $f(x, t) = x - F(x, t)$, where $F(X, I)$ is relatively compact in E, $f(x, 0) = x$ and $f(X, I) \subset Y$. Then, if y_0 belongs to a bounded component of $E \setminus X$ and to the unbounded component of $E \setminus Y$, it is proved that y_0 is swept by f in the sense that there is at least one solution of $y_0 = f(x, t)$. No homology theory is used in the proof of this theorem, which extends to Banach spaces a theorem proved by K. Borsuk [Monatsh. Math. **38** (1931), 381–386] for E_n. As an application the following generalization of a theorem of G. D. Birkhoff and O. D. Kellogg [Trans. Amer. Math. Soc. **23** (1922), 96–115] is proved in an elementary way. Let F_0 be a completely continuous map of the boundary X_0 of a bounded domain X in E with $0 \in X$ and such that $\|F_0(x)\| > \alpha > 0$ for x in X_0. Then F_0 has an invariant direction in the sense that there is at least one solution (x, μ) of the equation $x = \mu F_0(x)$.

R. G. Bartle (Urbana, Ill.)

12419:

Altman, M. A fixed point theorem in relative Banach spaces. Scripta Math. **25** (1960), 197–199.

Let B be the unit ball and S the unit sphere of a reflexive Banach space E admitting a Schauder basis. Let T be a weakly closed or weakly continuous operator on B to a bounded subset of E with $Tx = -T(-x)$ for $x \in S$. Then T admits a fixed point. {The term "relative" in the title, as well as the term "reflective" in the text, should be "reflexive".} *D. G. Bourgin* (Urbana, Ill.)

12420:

Nemyckiĭ, V. V. A method for finding all solutions of non-linear operator equations. Dokl. Akad. Nauk SSSR **131** (1960), 746–747 (Russian); translated as Soviet Math. Dokl. **1**, 330–331.

Let K be a convex compact subset of a Banach space E. Let f be a completely continuous self-map of E with $f(K)=B\subset K$. By uniform continuity, for some positive η_n inferior to 2^{-n}, f maps the finite η_n net, $N(n)$, of K onto the 2^{-n} net, $M(n)$, of B. The fixed points are obtained as limit points of sequences of pairs (y_n, fy_n), where y_n is a vertex of $N(n)$, according to the practical selection rule, $d(y_n, f(y_n))<2^{n-1}$. *D. G. Bourgin* (Urbana, Ill.)

12421:

Slugin, S. N. A linear semiordered topological space. Dokl. Akad. Nauk SSSR **131** (1960), 1261–1263 (Russian); translated as Soviet Math. Dokl. **1**, 426–428.

In preceding papers [same Dokl. **127** (1959), 34–36; MR **21** #7417] the author has developed algorithms for monotone approximation from above and below to the solutions of certain non-linear operator equations in K-spaces (complete vector lattices). The convergence of his approximations is order-convergence, except when the K-spaces are normed, in which case he can obtain topological convergence.

In the present communication (no proofs) he is interested in obtaining topological convergence for vector lattices where he does not have a norm. He considers, in a K-space X, a collection A of elements satisfying: (1) $u>0$ for all $u\in A$; (2) $u\wedge v\in A$ for all $u, v\in A$; (3) $(1/n)u\in A$ for all n and all $u\in A$. He then defines a topology on X by taking as neighborhoods of 0, the family of sets $\{G_u | u\in A\}$ where $G_u=\{x| \ |x|<u\}$. {Remark: Unless the reviewer has missed something, the resulting "topological" vector space in general lacks the property that neighborhoods of 0 are absorbing.}

The principal previous results of the author are now stated to hold for his topologized vector lattices, with order-convergence replaced by topological convergence, and in some cases, with a sharpening and simplifying of the results. *S. Kaplan* (Detroit, Mich.)

CALCULUS OF VARIATIONS

12422:

Vinti, Calogero. L'integrale multiplo del Calcolo delle Variazioni ed il problema dell'approssimazione delle funzioni. Ann. Mat. Pura Appl. (4) **52** (1960), 11–25.

The author proves the following principal theorem. Suppose that $f(x, z, p)$ $(x=(x_1, \cdots, x_r),\ p=(p_1, \cdots, p_r))$ is convex in all the $(2r+1)$ indicated variables on the set A of all (x, z, p) for which $x\in D'$, a convex domain, and suppose z is of class $H_\lambda^1(G)$ (W_λ^1 according to Sobolev, P_λ or B_λ according to Morrey, etc.) for some domain $G\subset D'$; then $I(z_h, R)\to I(z, R)$ on any bounded domain R with closure in G, where

$$I(z, R)=\int_R f[x, z(x), z_{x^1}(x), \cdots, z_{x^r}(x)]\,dx,$$

the right side being $+\infty$ in case the integrand is not summable, and $z_h(x)$ denotes the average of $z(\xi)$ for ξ in the cube of side $2h$ and center at x. The author also proves some minor refinements. The above theorem is a generalization of a theorem of Cinquini [Ann. Scuola Norm. Sup. Pisa (2) **4** (1935), 85–103] in which this result is proved for $r=2$ and z absolutely continuous in the sense of Tonelli. *C. B. Morrey, Jr.* (Berkeley, Calif.)

12423:

Picone, Mauro. Nuovi criteri sufficienti in un classico problema di calcolo delle variazioni. Ann. Mat. Pura Appl. (4) **53** (1961), 119–137.

The author has previously published a very general sufficiency theorem for multiple-integral problems of the calculus of variations [Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. **94** (1959/60), 341–352; MR **22** #9884]. Here he gives various equivalent or special forms of the sufficient conditions, and deduces some uniqueness theorems for boundary problems of Dirichlet type for certain classes of elliptic equations. There are a number of misprints. *L. M. Graves* (Chicago, Ill.)

12424:

Morrey, Charles B., Jr. Existence and differentiability theorems for variational problems for multiple integrals. Partial differential equations and continuum mechanics, pp. 241–270. Univ. of Wisconsin Press, Madison, Wis., 1961.

In the present paper the author gives final results concerning existence and differentiability properties of the solutions of variational problems for nonparametric regular multiple integrals. If G is a given bounded domain in the x-space E_n, $x=(x_1, \cdots, x_n)$, if $f(x, z, p)$ is a given function of x, z, p, for $x\in G$, $-\infty<z<+\infty$, and $p=(p_1, \cdots, p_n)$, $-\infty<p_i<+\infty$, if H is a convenient class of functions $z(x)$, $x\in G$, assuming given values on the boundary ∂G of G, the question is as to whether the integral $I[z]=(G)\int f(x, z(x), z_x(x))dx$ attains a minimum $z_0(x)$ in H, and whether $z_0(x)$ has satisfactory smoothness and differentiability properties, and satisfies the Euler equation for $I[z]$. These questions are answered in the affirmative in the present paper under the conditions below.

For H the author takes the class $H=H_{2k}^1(G)$, $k>\frac{1}{2}$, of equivalent classes of functions $z(x)$, $x\in G$, he has already investigated in previous papers, particularly in Univ. of California Publ. Math. (N.S.) **1** (1943), 1–130 [MR **6**, 180]. Namely, $z(x)$, $x\in G$, is a representative of an equivalent class $Z\in H$, if (a) $z(x)$ is measurable in G, (b) $z(x)$ is L^{2k} integrable in G, (c) there are n functions $p_i(x)$, $x\in G$, $i=1, \cdots, n$, satisfying (a) and (b) such that $(G)\int g(x)p_i(x)dx=-(G)\int z(x)(\partial g/\partial x_i)dx$ for every function $g(x)$, $x\in E_n$, of class C' and support K_g contained in the interior of G. Actually, the functions $p_i(x)$ are defined up to a null function independently of the representative $z(x)$ of the equivalent class Z of H to which $z(x)$ belongs. Briefly, the functions $p_i(x)$, $x\in G$, $i=1, \cdots, n$, can be thought of as the generalized partial derivatives of the function $z(x)$, and $z(x)\in H$, $z_x(x)=(p_1, \cdots, p_n)$. The boundary constraint is assumed to mean that $z=z^*$ on G where z^* is a given element of H. The function $f(x, z, p)$ is supposed to be of class C^2 in its $2n+1$ arguments and to satisfy either of the following two sets of conditions: (I) $mV^{2k}-K\leq f(x, z, p)\leq MV^{2k}$, $0<m\leq M$, $k>\frac{1}{2}$, $V=1+z^2+|p|^2$, $\sum(f_{p_i}^2+f_{p_ix_j}^2+f_z^2+f_{zx_j}^2)\leq M_1V^{2k-1}$, $\sum(f_{p_iz}^2+f_{zz}^2)$

$\leqq m_1 V^{2k-2}$, for all $x \in G_1, z, p$; or (II) $mV^k - K \leqq f(x,z,p) \leqq MV^k$, $0 < m \leqq M$, $k \geqq \frac{1}{2}n$, $V = 1 + |p|^2$, $\sum (f_{p_i}^2 + f_{p_i x}^2 + f_{p_i z}^2) \leqq M_1 V^{2k-1}$, $\sum (f_z^2 + f_{zz}^2 + f_{zx_i}^2) \leqq m_1 V^{2k}$. In either case, f is supposed to satisfy the regularity condition $mV^{k-1}|\lambda|^2 \leqq \sum f_{p_i p_j} \lambda_i \lambda_j \leqq MV^{k-1}|\lambda|^2$, for all $x \in G, z, p, \lambda = (\lambda_1, \cdots, \lambda_n)$, and the corresponding assumptions for k and V in (I) or (II).

The main result is that, under either conditions (I) or (II), there exists a function $z_0(x) \in H$, for which $I[z]$ has a minimum value (absolute minimum), z_0 satisfies the Euler equation, and z_0 is of class C^2 in each domain G_0 whose closure is in the interior of G. Also, if f is of some class C^r, or C^∞, or analytic, then z_0 is of class C^r ($r \geqq 2$), or C^∞, or analytic in each G_0.

Existence and differentiability theorems for multiple integral problems with $n=2$, $k>1$, had been given by L. Tonelli [Ann. Scuola Norm. Sup. Pisa (2) **2** (1933), 89–130] by the direct method of calculus of variations based on the concept of lower semicontinuity, and by the use of ACT functions and a uniform topology. In 1943, the author made the remark that for nonparametric multiple problems of calculus of variations it was advantageous to use classes of functions analogous to H (i.e., dropping the requirement of continuity of the ACT functions), with consequent need for L-topologies, and related refinements in the method. The author proved, among others, existence and differentiability theorems for $n=2$, $k \geqq 1$ [loc. cit.], and applied his results to the plateau problem in Riemannian manifolds [C. B. Morrey, Ann. of Math. (2) **49** (1948), 807–851; MR **10**, 259], and to other problems. Further remarks were made by G. Sigalov [Dokl. Akad. Nauk. SSSR (N.S.) **70** (1950), 769–772; MR **11**, 603], E. DeGiorgi [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43; MR **20** #172], J. Nash [Amer. J. Math. **80** (1958), 931–954; MR **20** #6592], J. Moser [to appear in Comm. Pure Appl. Math., 1961]. The present paper combines and improves all these results and remarks. A device often used is to minimize $I[z]$ in subclasses of functions $z(x) \in H$ for which $J[z] \leqq K$, where J is a convenient auxiliary functional and K a convenient constant.

L. Cesari (Ann Arbor, Mich.)

12425:

Kérimov, M. K. Sur la théorie des problèmes variationnels discontinus aux extrémités variables dans l'espace. Czechoslovak Math. J. **11** (**86**) (1961), 1–23. (Russian summary)

In the present paper the author studies the discontinuous variational problem for nonparametric curves in $(n+1)$-space E_{n+1} with both ends variable. If E_{n+1} denotes the $(n+1)$-space $(x, y) = (x, y_1, \cdots, y_n)$, and R a simply connected domain in E_{n+1}, it is assumed that three n-dimensional parametric varieties M_1, M_0, M_2 of class C^3 are given, M_0 dividing R into two parts R^-, R^+, $M_1 \subset R^-$, $M_2 \subset R^+$. Then the class C of all continuous curves C_{102} is considered which are the union of two arcs C^-, C^+, $C^- \subset R^-$, $C^+ \subset R^+$, where C^- joins a variable point 1 on M_1 to a variable point 0 on M_0, and C^+ joins the same point 0 to a point 2 on M_2, and where each of C^- and C^+ is a union of finitely many regular arcs. Two functions $F_1(x, y, p)$, $F_2(x, y, p)$ are given ($y = (y_1, \cdots, y_n)$, $p = (p_1, \cdots, p_n)$), both of class C^4, for $-\infty < p_j < +\infty$ ($j = 1, \cdots, n$), and $(x, y) \in R^-$ and $(x, y) \in R^+$ respectively. The author studies the functional $J = \int_{C^-} F_1(x, y, y')dx + \int_{C^+} F_2(x, y, y')dx$, in the class $\{C\}$ of curves C_{102} joining M_1 and M_2 and crossing once M_0 at an angular point 0. The methods and notations of G. A. Bliss [*Calculus of variations*, Univ. of Chicago Press, Chicago, Ill., 1946; MR **8**, 212] are used, as well as the quadratic forms of M. Morse's theory [*The calculus of variations in the large*, Amer. Math. Soc. Coll. Publ., Vol. 18, Amer. Math. Soc., New York, 1934]. The author establishes Euler, Weierstrass and Legendre conditions, the conditions of transversality and of discontinuity. The condition of Jacobi is formulated in terms of a family of extremals having an angular point on M_0. Also, the second variation is studied for the same extremals, and the corresponding proof of the Jacobi condition is given. An analogous study for $n=2$ was made by the author in a series of previous papers [Dokl. Akad. Nauk SSSR **79** (1951), 565–568, 719–722; **84** (1952), 213–216; Azerbaĭdžan. Gos. Univ. Uč. Zap. **5** (1958), 17–23; MR **13**, 474; **14**, 291; **20** #4202].

L. Cesari (Ann Arbor, Mich.)

GEOMETRY

See also 12031, 12057, 12102, 12465, 12501.

12426:

Погорелов, А. В. [Pogorelov, A. V.]. ★Лекции по основаниям геометрии [Lectures on the foundations of geometry]. Izdat. Har'kov. Gos. Univ., Kharkov, 1959. 134 pp. 6.25 r.

Table of contents: 1. An historical sketch of the foundations of geometry. 2. The modern axiomatic construction of Euclidean geometry. 3. A study of the axioms of Euclidean geometry. 4. Lobatchevskian geometry.

12427:

Mandan, Sahib Ram. Uni- and demi-orthocentric simplexes. J. Indian Math. Soc. (N.S.) **23** (1959), 169–184 (1961).

In this paper the author continues his program of generalizing the notion of the orthocenter of a triangle. For some preliminary work along the same lines, see Otto Dunkel's "Editorial Note" on Problem 4049 in the Amer. Math. Monthly **50** (1943), 577–578.

H. S. M. Coxeter (Toronto)

12428:

Lipman, Joe. A generalization of Ceva's theorem. Amer. Math. Monthly **67** (1960), 162–163.

Le théorème de Ceva est relatif à la figure formée par un triangle $A_1A_2A_3$ et un point P de son plan. La généralisation proposée est relative à un polygone plan $A_1A_2 \cdots A_n$ et $\frac{1}{2}d(d+3)-1$ points (P) dans son plan. Soit Q_r l'unique courbe de degré d qui passe par les points P et le sommet A_r; on désigne par $A^s_{r(s,s+1)}$ le produit des mesures algébriques des segments joignant A_s aux d points de rencontre de Q_r avec le côté A_sA_{s+1}. Alors

$$\prod_{r=1}^{n} \prod_{s=r+1}^{r+n-2} \frac{A^s_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = (-1)^n.$$

M. Decuyper (Lille)

12429:

Gerriets, C. J.; Pólya, G. On the location of the centroid of certain solids. Amer. Math. Monthly **66** (1959), 875–879.

The authors define the so-called pyramidoid: A pyramidoid is a polyhedron connected with a sphere; the center of the sphere is a special vertex of the pyramidoid called its apex; those faces of the pyramidoid of which the apex is not a vertex are tangent to the sphere and form jointly the base of the pyramidoid. Let O be the apex of the pyramidoid, C_V the centroid of its volume, C_B the centroid of its base. Both mass distributions involved are understood to be uniform.

The authors prove in an elementary way the following theorem: The points O, C_V and C_B lie on the same straight line, with C_V between O and C_B, and $OC_V = 3C_VC_B$. They formulate an analogous theorem also for the trianguloid.

L. Gyarmathi (Debrecen)

12430:

Dekker, Theodorus Jozef. ★Paradoxical decompositions of sets and spaces. Dissertation. Drukkerij Wed. G. van Soest, Amsterdam, 1958. 46 pp.

The author discusses and extends the many interesting results he and others (Mycielski, de Groot, etc.) obtained about paradoxical decompositions. Some of the older classical results are also discussed, and the paper contains a fairly complete bibliography on this subject.

P. Erdös (Budapest)

12431:

Segre, Beniamino. L'unirationalité de certaines variétés algébriques du point de vue de la géométrie différentielle et de la topologie. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 27–28. Centre Belge Rech. Math., Louvain, 1960.

A preliminary report on #12432 below.

12432:

Segre, Beniamino. Variazione continua ed omotopia in geometria algebrica. Ann. Mat. Pura Appl. (4) **50** (1960), 149–186.

Consider a continuous system of irreducible algebraic varieties V_k in an S_r. Define a "continuous variation" of varieties V_k by a path on the algebraic variety representing the system. To an arbitrary continuous variation of an irreducible, non-singular V_k in S_r, such that V_k remains non-singular at all intermediate positions, the author associates a homeomorphism between the corresponding images of these V_k's on the Riemannian of S_r [see also B. Segre, Ann. Mat. Pura Appl. (4) **43** (1957), 1–23; MR **19**, 579]. It follows that the superficial irregularity of an irreducible non-singular V_k is invariant for such continuous variations. Inequalities are also found relating the diverse numerical invariants of a generic non-singular V_k and those of its specializations obtained by continuous variation.

Turning to the fundamental group of an algebraic variety the author reestablishes the results of Chow [Amer. J. Math. **74** (1952), 726–736; MR **13**, 981], considers algebraic varieties which are coverings of other algebraic varieties, and deduces consequences for the invertibility of rational transformations between varieties under some special conditions concerning the exceptional subvarieties of the transformation.

An interesting application is given to entire Cremona transformations: Let a transformation of affine n-space be given by $\bar{x}_i = f_i(x_1, \cdots, x_n)$, $i = 1, 2, \cdots, n$, where the f_i are polynomials in $x_1, \cdots, x_n$. Then the necessary and sufficient condition for the existence of an inverse $x_j = g_j(\bar{x}_1, \cdots, \bar{x}_n)$ with the g_j polynomials in $\bar{x}_1, \cdots, \bar{x}_n$ is

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} = \text{const} \neq 0.$$

Finally, a section is devoted to extending Morin's work on unirational varieties [see U. Morin, Rend. Sem. Mat. Univ. Padova **21** (1952), 406–409; MR **16**, 615]. It is shown, inter alia, that there exist in S_4 varieties V_3^4 which are non-singular and unirational.

A. Gutwirth (Haifa)

12433:

Andreatta, Antonio. Sistemi lineari ∞^r reali, di ipersuperficie algebriche di S_r, con jacobiana priva di punti reali. Boll. Un. Mat. Ital. (3) **15** (1960), 134–139. (English summary)

Let $\varphi_0(x), \varphi_1(x), \cdots, \varphi_r(x)$ be $r+1 \geq 3$ forms of order n, in $r+1$ variables $(x) = (x_0, x_1, \cdots, x_r)$, with real coefficients; and suppose that their Jacobian $J(x) = \partial(\varphi)/\partial(x)$ has no real zero, $(x) \neq (0)$. Then, by means of topological considerations (similar to those developed by B. Segre [#12432]), it is proved that the equations $y_0 : y_1 : \cdots : y_r = \varphi_0(x) : \varphi_1(x) : \cdots : \varphi_r(x)$ have one and only one real solution $(x) \neq 0$, determined but for a proportionality factor, when a real $(y) \neq (0)$ is arbitrarily prescribed.

Hence it is deduced that the above assumption for $J(x)$ implies that n must be odd. Conversely, it is shown that—for each odd n—it is possible to choose the forms $\varphi_0(x)$, $\varphi_1(x), \cdots, \varphi_r(x)$ in such a way that $J(x)$ satisfies that assumption.

B. Segre (Rome)

12434:

Lozgačev, V. I. The determination of solid angles. Ž. Tehn. Fiz. **30** (1960), 1109–1114 (Russian); translated as Soviet Physics. Tech. Phys. **5** (1961), 1039–1044.

12435:

Spampinato, Nicolò. Sulla rappresentazione finita di elementi differenziali dell' S_r complesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 16–30.

This is an attempt to represent by various other structures, supposed to be simpler, or at any rate non-differential in character, the aggregates of curve elements E_1 and E_2 of projective space S_r. (An E_1 is a pair of consecutive points on a curve, or the combination of point and tangent; an E_2 is three consecutive points, or the combination of point, tangent, osculating plane, and curvature.) The author seems to be ignorant of most current work on this subject, and particularly of the definitive study by C. Longo [Rend. Mat. e Appl. (5) **13** (1955), 335–372; MR **17**, 1240].

An E_1 is mapped by the plane joining the images of its point P and its tangent t in two mutually skew S_r's in S_{2r+1}, both projectively related to the S_r in which the E_1 is given. These planes form a congruence of index 1 in S_{2r+1}. Similarly, the pencil of E_2's with a given point P, tangent t, and osculating plane π, is mapped on the S_5 joining the images of P, t, and π in three mutually skew

S_r's in S_{3r+2}, and these S_5's form a congruence of index 1 in S_{3r+2}. Within each S_5, the individual E_2's are mapped, somewhat awkwardly, by quadrics of a pencil. The mapping presents anomalies in both the special cases, of cuspidal and inflected E_2's. *P. Du Val* (London)

12436:

Spampinato, Nicolò. Sulla rappresentazione finita di elementi differenziali dell' S_r complesso. II. Rend. Accad. Sci. Fis. Mat. Napoli. (4) **26** (1959), 186–188. (English summary)

This second paper [see #12435] points out that in the mapping of E_1's by planes in S_{2r+1}, the planes representing E_1's at a fixed point and with tangents in a fixed plane form a pencil in S_3, and that the S_3's so arising form likewise a congruence of index 1 in S_{2r-1}.

P. Du Val (London)

12437:

Martinenko, V. S. On a property of the polyconic of a straight line in respect to a curve of the third order. Dopovidi Akad. Nauk Ukraïn. RSR **1961**, 150–152. (Ukrainian. Russian and English summaries)

Author's summary: "The following theorem is proved: the points of the intersection of a straight line with a curve of the third order form with the points of intersection of this straight line with its polyconic in respect to the third-order curve an equianharmonic relation."

12438:

Godeaux, Lucien. Une surface bicanonique de l'espace à quatre dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 113–118.

The cyclic homography in S_4

$$x'_0 : x'_1 : x'_2 : x'_3 : x'_4 = x_0 : \varepsilon x_1 ; \varepsilon x_2 : \varepsilon^2 x_3 : \varepsilon^2 x_4$$

leaves invariant every cubic of the form

$$x_0(a_{00}x_0^2 + a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4) + \alpha(x_1, x_2) + \beta(x_3, x_4) = 0,$$

where α, β are cubic forms, and ε a primitive cube root of unity. The surface F of intersection of two general such cubics belongs to the cyclic involution I generated by the transformation, and contains no united points. It is the canonical model of a regular surface of genera $p_g = p_a = 5$, $p^{(1)} = 10$. The canonical system has three subsystems belonging to the involution, namely, the isolated curve K_0: $x_0 = 0$, and two pencils $|K_1|$, $|K_2|$: $\lambda_1 x_1 - \lambda_2 x_2 = 0$, $\lambda_3 x_3 + \lambda_4 x_4 = 0$, respectively.

A model F' of the involution I on F is a regular surface of genera $p_g = p_a = 1$, $p^{(1)} = 4$; the bicanonical model is constructed as projective model of the sections of F by a system of quadrics belonging to the involution. The unique canonical curve on F' is the image K'_0 of K_0, and those of K_1, K_2 satisfy $|3K'_0| = |3K'_1| = |3K'_2| = K'_0 - K'_1 - K'_2$. The surface thus has a division group of order 3, the two zero divisors being $|K'_1 - K'_2| = |K'_2 - K'_0| = |K'_0 - K'_1|$, and the negative of this (which is also its double). *P. Du Val* (London)

12439:

Andreatta, Antonio. Piani grafici algebrici reali, desarguesiani o meno. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 350–356.

On appelle plan graphique algébrique réel ϑ un plan graphique dont les points sont les points d'un plan projectif ordinaire π, et dont les droites sont des courbes algébriques réelles de π, constituant un système σ algébrique réel (il faut faire abstraction des points réels isolés de ces courbes, et des courbes réelles isolées de ces systèmes). Le système σ est appelé système graphique de π; il est appelé desarguésien (non desarguésien) si ϑ est desarguésien (non desarguésien). L'auteur prouve d'abord que, si nous donnons, dans deux plans ordinaires projectifs réels π et π', deux systèmes graphiques desarguésiens σ et σ', il y a un homéomorphisme algébrique de π sur π' qui applique σ sur σ'. Il prouve encore que, donnés deux nombres entiers p et q ($p \geqq 0, q \geqq 0$) il y a toujours au moins un système graphique desarguésien et un système graphique non desarguésien, qui sont les parties réelles de deux systèmes algébriques ayant le premier genre égal à p, le deuxième égal à q (on appelle premier genre d'un système algébrique $\Sigma\infty^2$, le genre de la courbe générique de Σ, et on appelle deuxième genre de Σ le genre du système ∞^1 formé par les courbes de Σ qui passent par un point générique).

B. Segre [Rev. Math. Pures Appl. **1** (1956), no. 3, 35–50; MR **20** #1055] et N. H. Kuiper [Nieuw Arch. Wisk. (3) **5** (1957), 19–24; MR **19**, 56] avaient déjà donné des examples de systèmes graphiques non desarguésiens.

G. Zappa (Florence)

12440:

Melzi, Giovanni. Varietà luoghi di spazi lineari con assegnate proprietà di incidenza in S_n. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 167–212.

The assigned incidence properties are formulated thus: in a projective space Π_n, to determine the most general ∞^μ-family of linear subspaces Π_p such that neighbouring spaces intersect in a Π_h (focal space) of given dimension h, $h < p$.

There are two families of spaces which play essential roles in the discussion, the focal spaces, and the "join-spaces", i.e., the subspaces of dimension $2p - h$ spanned by pairs of Π_p through a focal space.

The problem is explained in terms of the representation of the family by points of an affine space Θ_μ; the family is characterized by sequences of indices μ_i, ν_j relating to the behaviour of the representations of the focal spaces and join-spaces in the neighbourhood of a point of Θ_μ.

Reduced Grassmann coordinates for the spaces Π_p are constructed effectively thus: take affine coordinates $y_1, \cdots, y_n$ and define Π_p by a set of $p+1$ points whose coordinates form the rows of a matrix $[A, B]$ where A is $p+1$ by $p+1$, and B is $p+1$ by $n-p-1$. Replace these points by the set $[1, A^{-1}B]$. The elements of $A^{-1}B$ are with some modifications the coordinates used. The problem is in this way reduced to one of the investigation of the solutions of a system of Pfaffian equations.

T. G. Room (Sydney)

12441:

Benedicty, Mario; Kay, David C. Projective classification of the plane trilinearities of dimension four. Rend. Sem. Mat. Univ. Padova **30** (1960), 101–123.

The trilinearities of the type investigated are shown to be reducible to three types: (i) $\rho z_k = \sum_{ijk} x_i y_j$ ($i, j, k = 1, 2, 3$),

say $\rho z = Ay$, where the x_i satisfy $\det A = 0$; (ii) $\rho z = Ay$, where all nine elements (quadratic in x_i) of adj A have a common linear factor; (iii) $\rho_i y_j = x_i z_k$. These are then shown to be classifiable, according to the speciality of A, into altogether eight types, all of which have simple forms of equations with no invariants. *T. G. Room* (Sydney)

12442:

Spencer, Jill C. D. On the Lenz-Barlotti classification of projective planes. Quart. J. Math. Oxford Ser. (2) **11** (1960), 241–257.

Se P è un punto ed l una retta di un piano grafico π, si dice che π è P-l transitivo se, comunque si prendano due punti X e X' di π distinti da P e non situati su l, esiste una P-l prospettività (cioè una collineazione per cui l è asse di punti uniti e P è centro di rette unite) che porta X in X'. Barlotti [Boll. Un. Mat. Ital. (3) **12** (1957), 212–226; MR **19**, 674], raffinando una precedente classificazione di Lenz, ha classificato i piani grafici in base alla configurazione F delle coppie punto-retta (P, l) rispetto alle quali il piano risulta P-l transitivo. Non si sa però se alcune classi di piani a priori possibili indicate da Barlotti siano vuote o no. L'autore prende in esame le classi II 3, III 2, II 1, I 2 di Barlotti, e prova, mediante un procedimento molto complesso, che la classe II 3 (cioè la classe dei piani per cui la configurazione F è formata dalle coppie (P, l) con P su una retta fissa r, l passante per un punto fisso R di r, e P ed l corrispondenti in una corrispondenza biunivoca fissa tra la punteggiata di sostegno r e il fascio di rette di centro R) è vuota. Ella prova poi che invece la classe III 2 (cioè la classe dei piani per cui la configurazione F è formata dalla coppia fissa (R, r), con R fuori di r, e da tutte le coppie (X, XR) per ogni X su r) non è vuota, poichè ad essa appartiene il noto piano di Moulton [Trans. Amer. Math. Soc. **3** (1902), 192–195]. Infine, vengono dati, senza dimostrazione, esempi di piani della classe I 2 (piani in cui F è formata da una sola coppia (P, l), con P non su l) e della classe II 1 (piani in cui F è formata da una sola coppia (P, l) con P su l). La non esistenza di piani della classe II 3 era stata provata, indipendentemente, da Pickert [Math. Z. **71** (1959), 99–108] sotto un'ipotesi restrittiva. *G. Zappa* (Florence)

12443:

Demaria, Davide Carlo. Sui piani grafici esagonali a caratteristica p. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 43–52.

The quadrangle (A_1, A_2, A_3, A_4) of the projective plane Q is said to be of characteristic p if and only if the points

$$P_0 = A_1,$$

$$P_i = A_1A_3 \cap A_4[A_2A_3 \cap P_{i-1}(A_1A_2 \cap A_3A_4)], \qquad i = 1, \cdots, p-1,$$

are all different, and $P_p = P_0$. If all quadrangles in Q have the same characteristic, then this integer is called the characteristic of Q. A projective plane of characteristic two is Pappian if it is finite [Gleason, Amer. J. Math. **78** (1956), 797–807; MR **18**, 593] or if it is generated by five points, three of which are collinear [Zaddach, Math. Z. **65** (1956), 353–388; MR **18**, 665]. The present author shows that if Q is hexagonal [that is, if the "Sechsecksbedingung" is satisfied; cf. Pickert, *Projektive Ebenen*, Springer, Berlin, 1955; MR **17**, 399; p. 54], then every quadrangle in Q has prime characteristic p and generates a subplane isomorphic to the Pappian plane over GF(p). The hexagonality hypothesis can be dropped if all quadrangles arising from a given (A_1, A_2, A_3, A_4) by permutations of the A_i have the same prime characteristic. In particular, every plane of characteristic three is hexagonal, and hence all its quadrangles generate subplanes isomorphic to the plane over GF(3). *P. Dembowski* (Frankfurt a. M.)

12444:

Cronheim, Arno. On Hermitian planes. Arch. Math. **11** (1960), 127–135.

Sei F_0 ein Körper von Charakteristik $\neq 2$, und V ein Vektorraum über F_0 mit $\dim V \geq 2$; es gebe einen Ring R mit folgenden Eigenschaften: $V \subseteq R$, $F_0 \subseteq$ Zentralisator von V, $A^2 \in F_0$ für alle $A \in V$. Dann wird V ein F_0-Vektorraum von Involutionen genannt. [Ein Beispiel bilden in einer euklidischen Koordinatenebene die Produkte der Spiegelungen an Geraden durch den Nullpunkt mit Streckungen aus dem Nullpunkt, d.h. die Matrizen $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.] Durch die zu der quadratischen Form $Q(A) = A^2$ gehörige symmetrische Bilinearform $\frac{1}{2}(AB + BA)$ wird V zu einem metrischen Vektorraum über F_0.

Der Verfasser betrachtet Translationsebenen über quadratischen Alternativkörpern F/F_0 von Charakteristik $\neq 2$; er nennt sie Hermitesche Ebenen. Die Translation, die 0, 0 in x, y überführt, sei mit (x, y) bezeichnet. Sei $\varepsilon \neq 0$ aus F_0. Sei $\alpha \in F_0$, $a \in F$, und $A(\alpha, a)$ der durch $(x, y)A(\alpha, a) = (x\alpha + y\bar{a}\varepsilon, xa - y\alpha)$ definierte Endomorphismus der Translationsgruppe ($a \to \bar{a}$ bezeichne den eindeutig bestimmten involutorischen Antiautomorphismus von F/F_0). Dann ist die Gesamtheit der $A(\alpha, a)$ ein F_0-Vektorraum V_s von Involutionen. Der Verfasser zeigt, dass sich unter den Translationsebenen die Hermiteschen Ebenen durch die Existenz eines solchen Vektorraumes von Involutionen mit gewissen zusätzlichen Eigenschaften kennzeichnen lassen.

Der Name "Hermitesche Ebenen" für die Translationsebenen über quadratischen Alternativkörpern F/F_0 von Charakteristik $\neq 2$ erklärt sich so: Sei X_m die aus den sämtlichen (x, xm) mit $m \in F$ bestehende Richtungsuntergruppe der Translationsgruppe, und sei für diese X_m eine Orthogonalität folgendermassen erklärt: $X_m \perp_s X_n$, wenn es in V_s eine Spiegelung S mit X_m als Achse und X_n als Zentrum gibt, das soll heissen, wenn es ein $S \in V_s$ gibt mit $\tau S = \tau$ für alle $\tau \in X_m$ und $\tau S = -\tau$ für alle $\tau \in X_n$. Dann wird diese Orthogonalität durch eine Hermitesche Form beschrieben: $X_m \perp_s X_n$ ist gleichwertig mit $1 + n\bar{m}\varepsilon = 0$. *F. Bachmann* (Kiel)

12445:

Vranceanu, Georges. Sur le plongement des espaces projectifs. C. R. Acad. Sci. Paris **251** (1960), 192–193.

The reviewer (and independently Kuiper) has described some embeddings of projective spaces (real, complex, quaternionic, Cayley) in euclidean spaces of various dimensions [see James, Proc. Cambridge Philos. Soc. **55** (1959), 294–298; MR **22** #236]. The present author replaces some of these by rational embeddings, without altering the dimensions involved. *I. M. James* (Oxford)

12446:

Böhm, Johannes. Untersuchung des Simplexinhaltes in Räumen konstanter Krümmung beliebiger Dimension. J. Reine Angew. Math. **202** (1959), 16–51.

To compute the area of a polygon one naturally dissects it into right-angled triangles. Analogously, to compute the content of any polyhedron or polytope, one dissects it into "orthoschemes" [L. Schläfli, *Gesammelte mathematische Abhandlungen*, I, Verlag Birkhäuser, Basel, 1950; MR **11**, 611]. An orthoscheme is a simplex whose bounding hyperplanes can be named in such an order that the angle between the rth and $(r+1)$th is (say) α_r while all non-consecutive pairs are at right angles. To express the "lower" angles in terms of these "dihedral" angles, the author uses the reviewer's notation [Bull. Calcutta Math. Soc. **28** (1936), 123–144; cf. pp. 126, 131] involving a set of four integers; e.g., $\alpha_r=[r-1, r, r+1, r+2]$. He considers, in particular, the case of an asymptotic orthoscheme (with one vertex at infinity) in hyperbolic space. He also shows how the trigonometrical rules of Napier and Engel can be extended from 2 to n dimensions. He points out that Schläfli's formula for the differential of the content of an orthoscheme is valid not only in spherical space but also in hyperbolic space [cf. Coxeter, *Non-Euclidean geometry*, 3rd ed., Univ. of Toronto Press, Toronto, 1957; MR **19**, 445; Chapter 15]. He notes that the content of an orthoscheme in hyperbolic 3-space is expressible as a linear combination of Lobachevsky functions $\Lambda_2(z)=-\int_0^z \log\cos t\,dt$ of the dihedral angles along with an "invariant" u which is a trigonometrical function of these angles. [This u is Lobachevsky's δ; see Coxeter, Quart. J. Math. Oxford Ser. **6** (1935), 13–29; cf. p. 23.] The function $\Lambda_2(z)$ is closely related to the "dilogarithm" $L_2(z)=-\int_0^z t^{-1}\log(1-t)\,dt$. P. Müller [Dissertation, Bonn, 1955] has extended these results to an expression for the content of an orthoscheme in hyperbolic 5-space in terms of the "trilogarithm" $L_3(z)=\int_0^z t^{-1}L_2(t)\,dt$. This provides a geometrical proof for Kummer's identity

$$L_3(e^{2i\alpha})-L_3(e^{-2i\alpha}) = \tfrac{2}{3}i\alpha(\alpha-\pi)(2\alpha-\pi).$$

The author finds a direct geometrical interpretation for the "invariant" $u=u^{(4)}$ as one of the angles in an asymptotic orthoscheme related in a simple manner to the given orthoscheme. He generalizes it from $u^{(4)}$ in hyperbolic 3-space to $u^{(2m)}$ in hyperbolic $(2m-1)$-space, and expresses it in terms of the reviewer's two-digit symbols (h, k) [Bull. Calcutta Math. Soc. **28** (1936), 123–144 (cf. p. 134); *Regular polytopes*, Methuen, London, 1948 (cf. p. 161); Proc. Internat. Congr. Math., Amsterdam, 1954, vol. III, pp. 155–169, Noordhoff, Groningen, 1956 (cf. p. 159); MR **10**, 261; **19**, 304]. In fact, he finds that

$$\tan^2 u^{(2m)} = -(0, 2m-1).$$

H. S. M. Coxeter (Toronto)

12447:

Ruben, Harold. On the geometrical moments of skew-regular simplices in hyperspherical space, with some applications in geometry and mathematical statistics. Acta Math. **103** (1960), 1–23.

In Euclidean N-space, N hyperplanes through the center of a unit (hyper-)sphere generally meet one another in N lines (diameters) and decompose the "surface" of the sphere into 2^N spherical simplexes. If one of these simplexes is regular, having all its dihedral angles β, all the simplexes are "skew-regular," having dihedral angles β and $\pi-\beta$. The author finds expressions for certain linear combinations of the contents of the n-dimensional elements of a skew-regular simplex. The motivation for this work appears in an earlier paper [Biometrika **41** (1954), 200–227; MR **16**, 153]. The present paper may fruitfully be compared with that of J. Böhm [#12446].

H. S. M. Coxeter (Toronto)

12448:

Böhm, Johannes. Inhaltsmessung im R_5 konstanter Krümmung. Arch. Math. **11** (1960), 298–309.

The volume of a quadrirectangular tetrahedron in hyperbolic space is known to be expressible in terms of the Lobachevsky function $\Lambda_2(z)=-\int_0^z \log\cos t\,dt$, applied to certain trigonometrical functions of the dihedral angles [the reviewer, Quart. J. Math. Oxford Ser. **6** (1935), 13–29]. Continuing the work of his earlier paper [#12446], the author expresses this result in a form capable of extension to spherical and hyperbolic spaces of more than three dimensions. He carries out the details for five dimensions, using the dilogarithm

$$L_2(z) = -\int_0^z t^{-1}\log(1-t)\,dt = \sum_1^\infty \frac{z^n}{n^2},$$

in terms of which

$$\Lambda_5(z) = z\log 2 - \tfrac{1}{2}i(z^2-\tfrac{1}{12}\pi^2)+\tfrac{1}{2}iL_2(-e^{-2iz}).$$

In fact, he expresses the content of the general five-dimensional orthoscheme as the integral of the sum of 22 terms, all of which are dilogarithms of different trigonometrical combinations of the dihedral angles.

H. S. M. Coxeter (Toronto)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 12111, 12501, B13353.

12449:

Hadwiger, H. Extremale konvexe Rotationskörper bei Aequator- und Meridiannebenbedingungen. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), 15–18. (Russian and Romanian summaries)

The author considers a convex solid of revolution (such as an egg). Every section through the axis is a curve of perimeter l and area f. The maximal section perpendicular to the axis is a circle of radius a. He establishes the inequalities

$$\tfrac{1}{2}\pi al \leqq F \leqq \pi a(l-2a), \quad \tfrac{1}{3}\pi af \leqq V \leqq \tfrac{1}{2}\pi af.$$

For instance, the sphere has $F=2al=\pi a(3l/\pi-2a)$ and $V=4af/3$. The upper bounds are attained only by cylinders, but the lower bounds are attained by a cone or any kind of double cone (symmetrical or asymmetrical).

H. S. M. Coxeter (Toronto)

DIFFERENTIAL GEOMETRY

See also 12163, B12807, B12812.

12450:

Савелов, А. А. [Savelov, A. A.] ★Плоские кривые: Систематика, свойства, применения [Plane curves:

Systematics, properties, applications]. Edited by A. P. Norden. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 293 pp. 9.40 r.

The subtitle is "Reference Handbook", and the book is not much more than that. Comparatively little space is devoted to general theory (for instance, birational transformations are only mentioned). Individual curves are treated in great detail, technical applications receive satisfactory attention, and applications to biology, physics, foundations of geometry, and even esthetics, are brought up. Historical notes accompany the discussion of individual curves. *G. Y. Rainich* (Notre Dame, Ind.)

12451:

Fiorio, Giovanni. Il prodotto scalare fra vettori sinusoidali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **94** (1959/60), 301–312.

L'A. esamina le proprietà comuni degli insiemi di coppie di vettori sinusoidali che dànno il medesimo prodotto scalare. *D. Gallarati* (Genoa)

12452:

Fiorio, Giovanni. Il prodotto vettoriale fra vettori sinusoidali. I. Proprietà generali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **94** (1959/60), 478–488.

L'A. indica alcune proprietà del prodotto vettoriale di due vettori sinusoidali, e discute la possibilità di rappresentare il prodotto in esame mediante un vettore complesso. *D. Gallarati* (Genoa)

12453:

Kahane, Arno. Sur les transformations où l'angle V, que la tangente à une courbe plane fait avec le rayon vecteur du point de contact, est un invariant. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 411–434. (Romanian. Russian and French summaries)

On s'occupe de la transformation conforme d'une courbe. Soit V l'angle que fait la tangente en le point M de la courbe γ avec le rayon vecteur du point M; la transformée de la courbe γ soit $\bar{\gamma}$. On étudie les transformations pour lesquelles $V=\bar{V}$. *A. Švec* (Prague)

12454:

Brauner, H. Die Strahlfläche 3. Grades mit konstantem Drall. Monatsh. Math. **64** (1960), 101–109.

It is known [see, for example, E. Müller and J. Krames, *Vorlesungen über darstellende Geometrie*, Bd. 3, Deuticke, Leipzig, 1931; p. 241] that the Cayley cubic surface is an example of a real algebraic ruled surface with constant torsion. In this paper it is shown that it is the only real algebraic ruled cubic surface with this property. *T. J. Willmore* (Liverpool)

12455:

Berezina, L. Congruences described by the axes of a tri-orthogonal trihedral. Latvijas PSR Zinātnu Akad. Vēstis **1959**, no. 10 (147), 71–76. (Russian. Latvian summary)

L'auteur étudie dans l'espace ordinaire les relations entre les congruences (r_1), (r_2), (r_3) décrites par les axes d'un trièdre trirectangle. Il en déduit diverses propositions en supposant d'abord que le sommet du trièdre décrit la surface moyenne d'une des congruences, puis que le trièdre est le trièdre canonique de (r_3) et enfin que la congruence (r_3) est isotrope. *M. Decuyper* (Lille)

12456:

Werner, Helmut. The existence of surfaces of constant mean curvature with arbitrary Jordan curves as assigned boundary. Proc. Amer. Math. Soc. **11** (1960), 63–70.

Riprendendo e sviluppando ulteriormente il risultato e i metodi già usati in un precedente lavoro [Math. Ann. **133** (1957), 303–319; MR **20** #1838], l'autore dimostra l'esistenza di una superficie di curvatura media costante H, avente per contorno una assegnata curva chiusa di Jordan, anche non rettificabile, contenuta nella sfera unitaria, nell'ipotesi che sia $|H|<\frac{1}{2}$. *E. Magenes* (Pavia)

12457:

Osserman, Robert. On the Gauss curvature of minimal surfaces. Trans. Amer. Math. Soc. **96** (1960), 115–128.

The author proves a number of theorems on minimal surfaces, notably on pieces of such surfaces whose spherical image omits the whole neighborhood of a point: a generalization of the situation for minimal surfaces with the representation $z=z(x, y)$, where the spherical image is contained in a semi-sphere. The main theorem may be stated as follows. Let p be a point of a minimal surface and N a simply-connected neighborhood of p. Let the distance along the surface from p to the boundary of N be at least d. Assume that for some $\alpha>0$ all normals to points of N make an angle of at least α with some fixed direction, and let the angle between this fixed direction and the normal at p be ω. Set $a=\cot(\omega/2)$, $r=\cot(\alpha/2)$. Then the Gaussian curvature K at p satisfies the inequality (1) $|K|\leq d^{-2}h^2(a, r)$. (Here $h(a, r)$ is a certain given function of a and r.) In the case $a=0$, when the normal at p is (opposite) parallel to the fixed direction, the inequality (1) reduces to $|K|\leq d^{-2}[2r(3+r^2)/3]^2$ which is sharp, as is shown by examples. Another special case is that in which the minimal surface has a representation $z=z(x, y)$, such that $\alpha=\pi/2$, $r=1$. Here $|K|\leq d^{-2}W^{-2}H^2(a)$, where W is the value at p of $(1+z_x^2+z_y^2)^{1/2}$ and certainly $H^2(a)<8$. If the neighborhood N lies over the disk $x^2+y^2<R^2$ with the point p projecting onto the origin, then $d\geq R$ and it follows that (2) $|K|\leq CR^{-2}W^{-2}$ with $C=8$. This is an improvement over previous estimates by E. Heinz, E. Hopf, and the reviewer (who had proved (2) with $C=12.25$) of the so-called Heinz-Hopf constant. The value $C=8$ is close to the best possible, but the latter is not yet known. Further topics include examples of complete simply-connected minimal surfaces with finite and with infinite total curvature and minimal surfaces whose spherical mapping is one-to-one.

Remark 1: The author has in the meantime [Comment. Math. Helv. **35** (1961), 65–76; MR **23** #A576] removed the assumption that the neighborhood N be simply-connected. Remark 2: A numerical computation by the reviewer and M. Stein shows that, in fact, $H^2(a)\leq 7.701$. *J. C. C. Nitsche* (Rio Piedras)

12458:

Terracini, Alessandro. Gli elementi curvilinei piani del

terz'ordine e una generalizzazione del teorema di Meusnier. Boll. Un. Mat. Ital. (3) **14** (1959), 66–77. (English summary)

Un elemento curvilineo piano del terzo ordine E_3 (cioè, nella terminologia di Ehresmann un getto infinitesimale di ordine 3 di R^2 in R (R rette, R^2 piano cartesiano reale)) che non sia di flesso individua un'ellisse di eccentricità minima che lo contiene. Tale ellisse viene detta ellisse di ipercurvatura; il suo centro, centro di ipercurvatura dell' E_3 considerato. L' E_3 è individuato dalla conoscenza del suo centro, della sua tangente e del suo centro di ipercurvatura.

Tali concetti generalizzano nel modo più naturale quelli di cerchio osculatore e centro di curvatura di un E_2 (non di flesso).

La nozione di centro di ipercurvatura dà luogo alla seguente generalizzazione di un ben noto teorema di Meusnier: Sia O un punto semplice di una superficie F, g una tangente non principale ad F in O; il luogo dei centri di ipercurvatura delle sezioni piane di F per g è un'ellisse contenuta in un piano per la tangente g' coniugata di g.

F. Gherardelli (Florence)

12459:

Scarafiotti, Anna Rosa. Proiezioni piane di coppie di elementi curvilinei spaziali. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 121–126.

Let $E_2(A)$, $E_2(B)$ be two curvilinear elements of the second order in a three-dimensional projective space through points A, B, respectively. A projective invariant I relative to the projections of the elements $E_2(A)$, $E_2(B)$ onto a general plane from two distinct centers U, V respectively, is obtained, and the loci of the centers U, V, of projections for a fixed I are discussed in each of the following cases: (1) one of U, V is fixed; (2) U, V are coincident; (3) U, V are in a given plane through A, B.

C.-C. Hsiung (Bethlehem, Pa.)

12460:

Fava, Franco. Sulla doppia proiezione di elementi curvilinei. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 127–136.

Let E be a curvilinear element in a three-dimensional projective space with tangent t at an ordinary point O, and E', $\bar{E}'$ its two projections from two centers S, $\bar{S}$ of projection, respectively, onto a given plane, which contains O but not S, $\bar{S}$ and t. There are obtained analytic conditions, together with geometric interpretations, for E', $\bar{E}'$ to have a contact at O up to the fourth order inclusive.

C.-C. Hsiung (Bethlehem, Pa.)

12461:

Kawaguchi, Akitsugu. Die Differentialgeometrie höherer Ordnung. IV. Erweiterung der verallgemeinerten Rheonomtransformation von Flächenelementen höherer Ordnung und R_K-Extensoren. Publ. Math. Debrecen **7** (1960), 256–276.

[The first three parts appeared in J. Fac. Sci. Hokkaido Univ. Ser. I **9** (1940), 1–152, 153–188; **10** (1941), 77–156; MR **2**, 22; **3**, 20; **9**, 605.] X_N and P_K are N- and K-dimensional manifolds whose point-coordinates are x^i ($i=1, 2, \cdots, N$) and u^α ($\alpha=1, 2, \cdots, K$) respectively. Every K-dimensional surface in X_N is given by $x^i=x^i(u^\alpha)$ ($N>K$) and at every point in P_K a K-dimensional surface element of order M is given by $(x^i, p_\alpha{}^i, \cdots, p^i_{\alpha(M)})$ ($p_\alpha{}^i=\partial x^i/\partial u^\alpha$, etc.). A system of $N\binom{K+M}{M}+K$ variables $(u^\alpha, x^i, p_\alpha{}^i, \cdots, p^i_{\alpha(M)})$ is called the coordinates of a P-expoint in the manifold $M_{(N,M,K)}$. In § 1 a group G_H is extended from the parameter transformation $u^\beta=u^\beta(u^\alpha)$, where $H=K\left\{\binom{K+M}{M}-1\right\}$. A contra- and covariant P_K-vector density and many other geometric quantities are defined. For a given contra- and covariant P_K-vector density many other P_K-vector densities of another kind are constructed. In § 2 the author considers a transformation of the form: $x^\alpha=x^\alpha(x^i, u^\alpha)$, $u^\beta=u^\beta(u^\alpha)$ in the product manifold $X_N\times P_K$ and calls it a K-tuple generalized rheonomic transformation or, in short, an R_K-transformation. He also considers a scleronomous transformation, $x^\alpha=x^\alpha(x^i)$, $u^\beta=u^\beta(u^\alpha)$, extended R_K-transformations and extended scleronomous transformations. He obtains many formulae and theorems analogous to those in § 1 and also a concept of R_K-extensors. Some of them have been obtained by Hombu and Suguri [Mem. Fac. Sci. Kyūsyū Univ. Ser. A **2** (1941), 67–90; MR **3**, 311] and by the reviewer [J. Fac. Sci. Hokkaido Univ. Ser. I **11** (1946), 1–37; MR **10**, 404]. *T. Ohkubo* (Kumamoto)

12462:

Liang, You-Dong. Some theorems on geometrical objects. Sci. Record (N.S.) **3** (1959), 525–530.

The reviewer has studied the following problem. Given a transformation group in an n-dimensional space, is it possible to regard this group as an invariance group of a certain linear geometric object whose transformation law under the change of coordinates is specified? [See K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957; MR **19**, 576.] In this paper the author proposes to solve the same problem for a general geometric object and to extend the space to a space with supporting elements. *K. Yano* (Seattle, Wash.)

12463:

Jankiewicz, C. La dérivée de Lie du comitant géométrique. Ann. Polon. Math. **7** (1960), 193–199.

The author first gives the formula for the Lie derivative of a general geometric object [K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957; MR **19**, 576]. Then he derives the formula which gives the Lie derivative of a (differential) geometric comitant of M geometric objects in terms of Lie derivatives of these M geometric objects. He also defines a Hamilton derivative and gives some identities containing Hamilton derivatives which reduce to an ordinary Hamilton derivative and Noether identities, respectively, when the object is a scalar density of weight $+1$.

K. Yano (Seattle, Wash.)

12464:

Willmore, T. J. Generalized torsional derivation. Séminaire C. Ehresmann, 1957/58, exp. no. 17, 12 pp. Faculté des Sciences de Paris, 1959.

Derivations D on the (algebra of) tensor fields over a (C^∞) manifold are determined by their action on the functions f and vector fields v and should, on those, satisfy the "Leibnitz Rule", in particular, $D(fv)=Df\otimes v+f\cdot Dv$.

A. G. Walker's tensorial derivative [C. R. Acad. Sci. Paris **245** (1957), 1213–1215; MR **19**, 680] associated with an almost-complex structure and its torsion $H=\frac{1}{2}[h,h]$ (for the bracket notation, cf. Frölicher and Nijenhuis, Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 338–359 [MR **18**, 569]) is thus defined by $Dv=\frac{1}{2}h[hv,H]-\frac{1}{2}[v,H]$. From this there arises a new differential concomitant of an almost-complex structure, namely, DH, a tensor once contravariant and 4 times covariant. Its skew-symmetric part, called by Walker the second torsion, is expressible in simpler form, as $[H,hH]$, or as $[h,H\curlywedge H]$. The tensor DH provides a partial counter-example to a conjecture of the reviewer, in that $[h,h]$ is not the only differential concomitant of a vector 1-form, provided, however, $hh=-I$.

The aim of the present paper is to extend the torsional derivative to cases where the conditions $hh=-I$ and $H=\frac{1}{2}[h,h]$ are no longer needed. Generalizations are obtained to cases where H is not the torsion but an arbitrary tensor, once contravariant, satisfying some algebraic identities [cf. also the reviewer's abstract, Amer. Math. Soc. Not. **5** (1958), 327], and to cases where $hh=-I$ is replaced by $hk+kh=-2I$, or by $\alpha h(lk-kl)+\beta k(lh-hl)=I$. [Cf. also the end of § 3 in the reviewer's Nederl. Akad. Wetensch. Proc. Ser. A **54** (1951), 200–212; MR **13**, 281.] *A. Nijenhuis* (Seattle, Wash.)

12465:

Takasu, Tsurusaburo. Extended non-Euclidean geometry. Proc. Japan. Acad. **36** (1960), 179–182.

The starting point is the extended equi-affine group $\bar{\xi}^i=a_m{}^i(\xi^p)\xi^m+a_0{}^i$, $|a_m{}^i|=1$, $a_0{}^i=\text{const}$, $i,m,\cdots=1,2,\cdots,n$ with inverse $x^i=\Omega_m{}^i(\xi^h)\xi^m+\Omega_0{}^i$ [see same Proc. **34** (1958), 471–476; MR **20** #7290]. Then the $(n+1)$ Π-geodesic $(n-1)$-flats $\alpha_m{}^i\xi^m+\alpha_0{}^i=0$, $i=1,2,\cdots,n+1$, $\alpha_m=c_i\Omega_m{}^i$, $\alpha_0=c_i\Omega_0{}^i+c_0$ (the c are constants), are introduced [Yokohama Math. J. **6** (1958), 89–176; MR **21** #5994], and the X^i defined as points in the equi-affine space by $\rho(\xi^h)X^i=a_m{}^i\xi^m+a_0{}^i$. Then one is led to the transformations $\rho\bar{X}^i=\alpha_j{}^iX^j$ $(|\alpha_j{}^i(X^k)|=1$, $\rho\bar{X}^i=\bar{\xi}^i-a_0{}^i)$, which underlie what is called an extended projective transformation group, of which the ordinary projective group is a subgroup. An extended Cayley-Klein representation is based on a Π-geodesic hyperquadric $X^iX^i=0$, an extended Poincaré-Klein representation on a Π-geodesic hypersphere $\xi^p\xi^p=k^2$, $p=1,2,\cdots,n$. The extended equiform transformation [see loc. cit., MR **21** #5994] $\bar{\xi}^p=2k^2\xi^p/(\xi^h\xi^h+k^2)$ is a mutual mapping of both representations and introduced as the extended Darboux-Liebmann transformation.

D. J. Struik (Cambridge, Mass.)

12466:

Villa, Mario. De la théorie des transformations ponctuelles à celle des variétés asymptotiques. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 55–63 (1959). (Romanian. French and Russian summaries)

12467:

Villa, Mario. Dalla teoria delle trasformazioni puntuali a quella delle varietà quasi-asintotiche. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 353–359.

This is an expository article describing results mostly due to the author, appearing in a series of papers written from 1939 to 1956. Let V be a differentiable manifold in euclidean or projective space. A quasi-asymptotic submanifold of type $\sigma_{r,s}$ is a submanifold such that at each $p\in\sigma_{r,s}$, the r-fold osculating linear space of V and the s-fold osculating linear space of $\sigma_{r,s}$ span a smaller linear space than for a generic submanifold through p. A correspondence between two linear spaces L_1, L_2 is a subvariety X of $L_1\times L_2$. Properties of the correspondence are expressed in terms of quasi-asymptotic properties of X in $L_1\times L_2$; e.g., if $L_1=L_2$ is a plane, and X is a surface, the correspondence fails to be regular at p if and only if X has the quasi-asymptotic property at p, and the various possible kinds of degeneracy are classified in terms of different types of quasi-asymptotic surfaces.

A. H. Wallace (Bloomington, Ind.)

12468:

Merza, J. Sur la courbure affine des courbes. Publ. Math. Debrecen **7** (1960), 65–71.

At a point P_0 of a curve in the affine plane the tangent parabola p is drawn. A point P of the curve is projected into a point E on p in the direction of the affine normal. When d is the affine distance of the two linear elements obtained in this way, s the affine length of arc P_0P, and e the affine length of arc P_0E, then the affine curvature at P_0 is $k=\lim(2d)^3/s^5=\lim(2d)^3/e^5$, $P\to P_0$. Corresponding formulas are derived for space, in the case of the first curvature with the aid of the hyperbolic paraboloid $3x_3=x_1x_2$, in that of the second curvature with the aid of the elliptic paraboloid $2x_2=x_1{}^2+x_3{}^2$ (equations expressed in the trihedron of Winternitz). Then the formulas are $k_1=-(20/\sqrt{3})\lim d/s^5$, $k_2'=30\lim d/s^5$, $P\to P_0$ (k_2' instead of k_2 because of dimension considerations).

D. J. Struik (Cambridge, Mass.)

12469:

Švec, Alois. Sur le problème de la stratification des congruences de droites. Czechoslovak Math. J. **10 (85)** (1960), 299–303. (Russian summary)

The author considers in a projective space S_5: (1) a congruence L_1 of ∞^2 lines which can be distributed in two different ways in developable surfaces (Laplace congruence); (2) a congruence $L_2(\infty^2)$ of 3-dimensional spaces; and a 1-1 correspondence between them. Stratifiability of L_1, L_2 is defined by the following conditions: (a) there is an ∞^2 system of surfaces V_2 such that through each point of a line of L_1 passes a single V_2, and the tangent plane at that point intersects the corresponding space in a line; (b) there is an ∞^2 system of surfaces W_2 such that through each point of a space of L_2 passes a single W_2, and the tangent plane at that point contains the corresponding line. It is proved that for a stratifiable pair L_1, L_2 this is the dual of a Lagrange congruence of lines L_2 and that the surfaces V_2, W_2 admit a conjugate system. If the tangent space along a line of the 3-manifold covered by L_1 is the corresponding space of L_2, the pair L_1, L_2 is called strongly stratifiable: the congruence L_1 has focal R-surfaces. *E. Bompiani* (Rome)

12470:

Gallarati, Dionisio. Una proprietà caratteristica della varietà cubica a tre dimensioni dello spazio S_5. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 187–200.

In questa Nota si dimostra il seguente teorema: Sia V_3 una varietà differenziabile di classe C^1 e dimensione 3 di uno spazio proiettivo complesso S_r, $r \geq 5$, non cono e non luogo di spazi S_3. Se gli S_3 tangenti a V_3 sono incidenti a quattro rette a tre a tre indipendenti, allora tali spazi tangenti incontrano ∞^2 rette (di V_3), e anzi la V_3 è una varietà cubica $S_2 - V_3^3$ (razionale, normale) di S_5.

Se poi V_3 è di classe C^2 la V_3^3 può essere caratterizzata da condizioni più semplici. *F. Gherardelli* (Florence)

12471:

Ostianu, N. M. On the geometry of an n-dimensional surface of a $(2n-1)$-dimensional projective space. Dokl. Akad. Nauk SSSR **136** (1961), 775–778 (Russian); translated as Soviet Math. Dokl. **2**, 131–134.

To a generic point M of an n-dimensional surface of a projective space of $N = 2n-1$ dimensions a reference system composed of $2n$ points $M_0, M_1, \cdots, M_{2n-1}$ is adjoint such that M_0 is the same as M and the other M_i are in the tangent space. A canonization is defined which corresponds to the referring of the surface to a system of asymptotic lines. Then n fields of quadrics are introduced, each passing through M and having a tangency of the second order with the surface in the direction $\omega^i = 0$ and a tangency of the third order along every asymptotic line $\omega^j = 0$ $(\hat{j} \neq \hat{i})$. Here ω^i $(i = 1, 2, \cdots, n)$ are the principal forms in the sense of S. P. Finikov, *Metod vnešnih form Kartana v differencial'noĭ geometrii* [OGIZ, Moscow-Leningrad, 1948; MR **11**, 597; p. 407], where also the secondary forms are defined. It is possible to connect, in an invariant way, with every asymptotic line ω^i a one-parametric pencil of such quadrics. These quadrics allow the definition of the two $(n-1)$-planes, which can be considered generalizations of the two edges of Green for $n = 2$. The notion of Darboux surfaces can also be generalized into a pencil of hyperquadrics. The paper is based on the investigations of G. F. Laptev, Trudy Moskov. Mat. Obšč. **2** (1953), 275–382 [MR **15**, 254].

D. J. Struik (Cambridge, Mass.)

12472:

Atanasyan, L. S. On the theory of rigged surfaces of a multidimensional projective space. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 3–44. (Russian)

12473:

Redozubova, O. S. Metric theory of pairs of T-congruences. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 135–175. (Russian)

12474:

Atanasyan, V. A. Construction of the intrinsic geometry of surfaces of a multidimensional affine space. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 177–217. (Russian)

12475:

Izmaĭlov, V. D. On the affine theory of two-dimensional surfaces. Moskov. Gos. Ped. Inst. Uč. Zap. **108** (1957), 219–259. (Russian)

12476:

Todorova, G. S. On Weyl's problem in a Lobačevskiĭ space. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1193–1196. (Ukrainian. Russian and English summaries)

Author's summary: "The author establishes the existence of a regular surface in a Lobachevsky space realizing the fourfold continuously differentiable metric of curvature of the principal curvature of the space, given on the sphere."

12477:

Yano, Kentaro. Conformal transformations in Riemannian and Hermitian spaces. Bull. Amer. Math. Soc. **66** (1960), 369–372.

The following generalization of a theorem of S. I. Goldberg [same Bull. **66** (1960), 54–58; MR **22** #11346] is proved: A conformal Killing vector v^h in an n-dimensional $(n > 2)$, compact, almost-Kählerian space defines an automorphism of the space.

C. B. Allendoerfer (Seattle, Wash.)

12478:

Ishihara, Shigeru. On infinitesimal concircular transformations. Kōdai Math. Sem. Rep. **12** (1960), 45–56.

An infinitesimal transformation v^h in a Riemannian space with metric tensor g_{ji} is said to be concircular if it carries any geodesic circle into a geodesic circle, a geodesic circle being defined as a curve satisfying

$$\frac{\delta^3 \xi^h}{ds^3} + \frac{d\xi^h}{ds} g_{ji} \frac{\delta^2 \xi^j}{ds^2} \frac{\delta^2 \xi^i}{ds^2} = 0$$

[K. Yano, Proc. Imp. Acad. Tokyo **16** (1940), 195–200, 354–360, 442–448, 505–511; **18** (1942), 446–451; MR **2**, 165, 303; **7**, 330].

The author first proves the following. In order that v^h be an infinitesimal concircular transformation (i.c.t.), it is necessary and sufficient that

$$\underset{v}{\mathfrak{L}}\, g_{ji} = 2\rho g_{ji}, \quad \nabla_j \rho_i = \phi g_{ji},$$

hold, ρ and ϕ being certain scalar functions. After having studied the local structure of a Riemannian space admitting an i.c.t., he next proves the following theorems. If a compact Riemannian manifold admits an i.c.t., then the manifold is a spherical space. If a compact Einstein space admits a non-homothetic i.c.t., then the space is a spherical space. Now consider a conformal circle in a Riemannian space which is defined by

$$\frac{\delta^3 \xi^h}{ds^3} + \frac{d\xi^h}{ds}\left(g_{ji} \frac{\delta^2 \xi^j}{ds^2} \frac{\delta^2 \xi^i}{ds^2} - \Pi_{ji} \frac{d\xi^j}{ds} \frac{d\xi^i}{ds} \right) + \Pi_i^h \frac{d\xi^i}{ds} = 0,$$

where

$$\Pi_{ji} = -\frac{K_{ji}}{n-2} + \frac{K g_{ji}}{2(n-1)(n-2)}, \quad \Pi_i^h = \Pi_{ia} g^{ah},$$

K_{ji} and K being the Ricci tensor and curvature scalar. The projective parameter t on it is defined by

$$\{t, s\} = \frac{1}{2} g_{ji} \frac{\delta^2 \xi^j}{ds^2} \frac{\delta^2 \xi^i}{ds^2} - \Pi_{ji} \frac{d\xi^j}{ds} \frac{d\xi^i}{ds},$$

where $\{t, s\}$ is the Schwarzian derivative of t with respect to s. If, for any point P on any conformal circle C, any projective parameter t of C vanishing at P takes any real value along the conformal circle C, the space is said to be

conformally complete. After this definition, the author proves the following theorems. Let M be a conformally complete Riemannian manifold having constant scalar curvature. If M admits an i.c.t., then M is a spherical space. Let M be a conformally complete Einstein space. If M admits a non-homothetic i.c.t., then M is a spherical space.

In the last section, the author states the following theorems. If a complete non-flat Riemannian manifold admits an i.c.t., then its local homogeneous holonomy group at any ordinary point ($\partial_i\rho \neq 0$) is the special orthogonal group $SO(n)$. If a non-flat conformally flat Riemannian manifold admits an i.c.t., then the local homogeneous holonomy group at any ordinary point is the group $SO(n)$. *K. Yano* (Seattle, Wash.)

12479:

Vranceanu, G. Le prolongement des connexions affines rationnelles dans P_n. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 95–102. Centre Belge Rech. Math., Louvain, 1959.

Supposons que $x^1, \cdots, x^n$ sont des coordonnées cartésiennes de l'espace euclidien E_n, qui, complété par les points à l'infini, nous donne P_n avec les voisinages: $V = E_n$; $V_1(x'^1, \cdots, x'^n)$, $x'^1 = (x^1)^{-1}$, $x'^h = x^h(x^1)^{-1}$ $(h > 1)$; et V_p $(p > 1)$, où le rôle de la variable x^1 est joué par x^p $(p > 1)$. Supposons que la connexion Γ_{jk}^i soit donné dans V et Γ_{jk}^i sont des fonctions rationnelles en x^i, alors l'auteur s'occupe du prolongement de la connexion dans les voisinages V_p. On résout la question analogue pour une métrique $ds^2 = a_{ij}dx^i dx^j$ (a_{ij} étant des fonctions rationnelles) donnée dans V. *A. Švec* (Prague)

12480:

Hicks, N. An example concerning affine connexions. Proc. Amer. Math. Soc. **11** (1960), 952–956.

In this note the author constructs a simply connected manifold on which is defined a complete affine connexion with the following property: for each integer $n > 0$ there are two points such that any broken geodesic between them must contain at least n breaks. This example shows that the main theorem of a previous paper [Illinois J. Math. **3** (1959), 242–254; MR **21** #6597] does not hold if a bound is placed on the number of breaks a broken geodesic may have. *T. J. Willmore* (Liverpool)

12481:

Haimovici, Adolf. Spazi a connessione affine equivalenti al piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 333–338.

Abstract of the paper reviewed below. *A. Kawaguchi* (Sapporo)

12482:

Haimovici, Adolf. Espaces à connexion affine équivalents avec le plan. II. Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Ști. Mat. **10** (1959), 283–305. (Romanian. Russian and French summaries)

Several sufficient conditions are obtained for a two-dimensional space with affine connection to be equivalent to a plane, i.e., the conditions for the existence of a system of solutions $\Gamma_{ij}^k(x^p)$ $(i, j, k, \cdots = 1, 2)$ satisfying the system of differential equations $R^l{}_{ijk} = T_{ij}^k = 0$, where $R^l{}_{ijk}$ and T_{ij}^k are the curvature tensor and the torsion tensor respectively. For example, if the functions Γ_{12}^i of class C^1 in a domain $\alpha \leq x^1$, $x^2 \leq \beta$ and $\varphi_{11}^i(x^1)$, $\varphi_{22}^i(x^2)$ of class C^1 in an interval $\alpha \leq x^1 \leq \beta$ or $\alpha \leq x^2 \leq \beta$, are given arbitrarily except for the conditions $|\Gamma_{12}^i| \leq A$, $|a_i| \leq B$, $|\varphi_{ii}^i(x^i)| \leq C$, where the a_i are formed of the Γ_{12}^i and their partial derivatives, and

$$A = \frac{\sigma}{2(\beta-\alpha)}, \quad B \leq \frac{\sigma^2 - 2\tau + 1}{4(\beta-\alpha)^2}, \quad C = \frac{\tau - \sigma}{2(\beta-\alpha)},$$

$$0 \leq \sigma \leq \tau \leq \tfrac{1}{2},$$

then there exists a unique system of functions Γ_{ij}^k of class C^1 in the domain, which satisfy the relations $R^l{}_{ijk} = T_{ij}^k = 0$ and $\Gamma_{11}^i(x^1, x_0^2) = \varphi_{11}^i(x^1)$, $\Gamma_{22}^i(x_0^1, x^2) = \varphi_{22}^i(x^2)$. *A. Kawaguchi* (Sapporo)

12483:

Willmore, T. J. Global theorems on manifolds which admit distributions. Séminaire C. Ehresmann, 1957/58, exp. no. 18, 7 pp. Faculté des Sciences de Paris, 1959.

This paper is a sequel to A. G. Walker [Quart. J. Math. Oxford Ser. **20** (1949), 135–145; (2) **6** (1955), 301–308; (2) **9** (1958), 221–231; MR **11**, 460; **19**, 312; **20** #6135] and T. J. Willmore [Proc. London Math. Soc. (3) **6** (1956), 191–204; Quart. J. Math. Oxford Ser. (2) **7** (1956), 269–276; MR **19**, 455; **20** #4299]. In Arts. 1–4, notations and identities of the $*$-operation are recapitulated. Two types of problems are considered in Arts. 5, 6 and 7. Art. 5 contains solutions of problems 1–5. Problem 1: To find an affine connection (L_{jk}^i) with the property that D' is parallel with respect to L and L is symmetric when D' is integrable. (Hereby M is an n-dimensional differentiable manifold of class ∞ which admits a C^∞-distribution of r-planes D'. D'' is a complementary distribution of $(n-r)$-planes.) Problem 2: To find an affine connection (L_{jk}^i) with the property that D', D'' are both parallel with respect to L, and L is symmetric when D', D'' are integrable. Problem 3: Given two supplementary distributions D', D'', to find a positive definite Riemannian metric with respect to which D' and D'' are orthogonal. Problem 4: For any complementary distributions D', D'', orthogonal with respect to a metric tensor g_{ij}, to find a global connection L such that D', D'' are parallel with respect to L, and also $g_{ij|k} = 0$. Problem 5: To find a set of necessary and sufficient conditions in order that a manifold M_n which admits a distribution D' can be given a Riemannian structure with respect to which D' is parallel. That the solutions given to problems 1, 2, 3, 4 are not unique is noted. Also, the following theorem is stated: If a compact orientable M_n admits a distribution D' of r dimensions, parallel with respect to a positive definite Riemannian metric, then necessarily $b_r > 0$, where b_r is the rth Betti number.

In Art. 6 Lie group manifolds are treated in relation to the distributions D_L, D_R giving rise to left- and right-translations respectively. Art. 7 consists of problems concerning the existence of distributions on homogeneous spaces which are compatible with the homogeneity of the space. The results are as follows. Theorem 1: If D_O is a subspace of the tangent space to M at O, then D_O generates a homogeneous distribution over M if and only if D_O is invariant under the linear isotropy group at O (i.e., fixing O). Theorem 2: No sphere S^n admits a distribution

homogeneous with respect to $SO(n)$. Theorem 3: If V is a linear subspace of A (the Lie algebra of the Lie group G) disjoint from the isotropy subgroup H of G, then V generates a homogeneous distribution over M if and only if $H \times V \subset H + V$. Theorem 4: The homogeneous distribution generated by V is integrable if and only if $H + V$ is a subalgebra of A. The following is stated: In order to find an integrable distribution over M it is merely necessary to find a subalgebra of A which contains H. If H' is such a subalgebra, we can write $H' = H + V$, $H \cap V = \emptyset$, and the subspace V thus determined will generate an integrable homogeneous distribution. *T. Takasu* (Yokohama)

12484:

Kimpara, Makoto. Sur les surfaces plongées dans un espace à connexion projective à quatre dimensions. Tensor (N.S.) **10** (1960), 61–72.

The main purpose of the present paper is to generalize the theory of surfaces in a projective space of four dimensions investigated by E. Bompiani [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) **5** (1927), 84–90, 143–149], G. Fubini [*Geometria proiettiva differenziale*, vol. 2, Bologna, 1927] and A. Kawaguchi [Japan. J. Math. **4** (1928), 275–305; **7** (1930), 267–273] to the case of a four-dimensional space with projective connection. On making use of Cartan's method of natural frames [*Leçons sur la théorie des espaces à connexion projective*, Gauthier-Villars, Paris, 1937; II partie, chap. I] and of the ideas of the above-mentioned authors, the author of this paper obtains several remarkable results and especially makes clear some relations between the properties of a surface and the curvature or torsion tensors of the space; e.g., if one of tangents to the lines of bitangency coincides with one of the tangents to the principal lines at every point of a surface in a space E_4, then E_4 has no torsion, and conversely, if E_4 has no torsion, the lines of bitangency of a surface always coincide with the principal lines of the surface. *A. Kawaguchi* (Sapporo)

12485:

Wong, Yung-Chow. Recurrent tensors on a linearly connected differentiable manifold. Trans. Amer. Math. Soc. **99** (1961), 325–341.

On a differentiable manifold M with a linear connection, a tensor field S is called recurrent if S is not identically zero and if the covariant differential ∇S is equal to the tensor product $S \otimes w$, where w is a certain 1-form on M. The main purpose of this paper is to study the properties of such tensor fields in terms of the bundle of frames B over M and obtain applications to the case of linear connections with recurrent torsion tensor or recurrent curvature tensor. The most interesting result contained here appears to be the following: The holonomy group of a linear connection with recurrent curvature tensor is at most of dimension $n(n-1)/2$, where n is the dimension of the manifold.

The author first gives an exposition of the basic concepts concerning linear connections, following S. S. Chern, *Differentiable manifolds* [mimeographed notes, Univ. of Chicago, Chicago, Ill., 1959] and also adopting from W. Ambrose and I. Singer [Trans. Amer. Math. Soc. **75** (1953), 428–443; MR **16**, 172] and K. Nomizu, *Lie groups and differential geometry* [Math. Soc. Japan, 1956; MR **18**, 821]. Then the author sets up a natural one-to-one correspondence between tensor fields of type (r, s) on M and sets of n^{r+s} functions on the bundle B of frames over M (with structure group $GL(n, R)$) which satisfy certain conditions. As the author observes as a remark, after doing this by the coordinate approach, this correspondence can be obtained intrinsically as follows. Let V_n be a real n-dimensional vector space with a fixed basis (e_i). Every point x of B, i.e., a frame (X_i) at a point $u \in M$, can be regarded as a linear isomorphism of V_n onto the tangent space $T_u(M)$ which maps each e_i upon X_i. The inverse mapping x^{-1} can be extended to an isomorphism of the (mixed) tensor algebra over $T_u(M)$ onto the (mixed) tensor algebra over V_n. Now for any tensor field S of type (r, s) on M, the components of the tensor $x^{-1} \cdot S_u$ with respect to (e_i) determine n^{r+s} functions $S^{ij\cdots}_{kl\cdots}(x)$ on B. This set of functions on B corresponding to S is transformed by the action of $a \in GL(n, R)$ on B in such a way as to reflect the transformation of tensors $x^{-1} \cdot S_u \to (xa)^{-1} \cdot S_u = a^{-1} \cdot (x^{-1} \cdot S_u)$ over V_n, which is expressed in terms of components with respect to (e_i). It is possible to determine the set of functions on B which corresponds to ∇S.

Using this correspondence, the author proves the following theorem. A tensor field S on M is recurrent if and only if the restrictions of the corresponding set of functions $S^{ij\cdots}_{kl\cdots}$ to any horizontal submanifold (submanifold of B consisting of all points which can be joined to any fixed point by a horizontal curve) have no common zero and are proportional to a set of constants. In particular, a recurrent tensor field has no zero point. As applications, the author characterizes a linear connection with recurrent torsion or recurrent curvature in terms of certain vector fields on B and proves the theorem on the dimension of the holonomy group which we quoted in the beginning. *K. Nomizu* (Providence, R.I.)

12486:

Hlavatý, Václav. The holonomy group. III. Metrisable spaces. J. Math. Mech. **9** (1960), 89–122.

This paper is a direct continuation of the previous papers with the same title by the author [same J. **8** (1959), 285–307, 597–622; MR **21** #899, 6002]. Let a space L_n with volume-preserving affine connection be given of which the curvature tensor is denoted by $L'_{\kappa\mu\lambda}$. Consider a symmetric tensor $g_{\lambda\nu}$ of rank n and the curvature tensor $R'_{\kappa\mu\lambda}$ formed by means of $g_{\lambda\nu}$. If $g_{\lambda\nu}$ is unknown and $L'_{\kappa\mu\lambda}$ is given, then $L'_{\kappa\mu\lambda} = R'_{\kappa\mu\lambda}$ represents a system of differential equations for $g_{\lambda\nu}$. The author solves the following problems: (1a) To find necessary and sufficient conditions for the system of differential equations $L'_{\kappa\mu\lambda} = R'_{\kappa\mu\lambda}$ to admit at least one solution $g_{\lambda\nu}$; (1b) To find the explicit form of at least one solution $g_{\lambda\nu}$, when these conditions are satisfied; (1c) To find the explicit form of all solutions, if one solution is given. That is, the problem (1a) is solved for any holonomy group and for any n in terms of a rank of a matrix, and in the case of a complete holonomy group it is solved in terms of the tensors which lead to the generators of the group. (1b) is solved for a complete holonomy group and any even n and for some other holonomy groups in the case of a particular n. (1c) is solved for the case of general, degenerate or complete holonomy groups. These problems may be summarized roughly as the problem: To find all metric

tensors having the same curvature tensors. This leads to the following problems: (2a) To find the explicit form of at least one covariantly constant quadratic symmetric tensor for a given symmetric connection; (2b) To find the explicit form of all solutions, when one solution of the problem (2a) is given. (2a) and (2b) are solved for the same cases as (1b) and (1c) respectively, and may be roughly summarized as follows: To find necessary and sufficient conditions in order that the connection parameters of a given symmetric connection reduce to Christoffel symbols, and to find all tensors which yield these Christoffel symbols. The following fact is noted. All solutions of the second system of problems are those of the first system of problems, but not conversely. The class of solutions of the first system contains, in some special cases, solutions which are not covariantly constant with respect to the connection with the given curvature tensor $L^{\nu}_{\omega\mu\lambda}$. This happens, for instance, in the case of a degenerate holonomy group. Both systems of problems are solved by means of purely algebraic methods.

A. Kawaguchi (Sapporo)

12487:

Hlavatý, Václav. The holonomy group. IV. The general L_n with symmetric connection. J. Math. Mech. 9 (1960), 453–496.

This paper is a continuation of the author's previous papers with the same title [same J. 8 (1959), 285–307, 597–622; MR **21** #899, 6002; #12486 above] and deals with a space L_n endowed with an affine symmetric connection which is not volume-preserving, i.e., $R_{\omega\mu} = R^{\alpha}_{\omega\mu\alpha} \neq 0$, where $R^{\nu}_{\omega\mu\lambda}$ is the curvature tensor of L_n. In the first part, which is an introduction to the second part, an L_n is defined in a different manner from the usual definition, and this definition enables one to stress the condition $R_{\omega\mu} \neq 0$ by means of a vector Q_μ and its alternative derivative $\partial_{[\omega}Q_{\mu]}$. The second part deals with the holonomy group $G_l(R)$ of an L_n from which the complete set of integrability conditions can be stated in a concise form. The condition $R_{\omega\mu} \neq 0$ plays a very important role for the structure of the holonomy group and under this condition there are two different types of holonomy groups. The group of both types contain at least two subgroups, one of them being always normal, i.e., invariant with respect to the transformations of the adjoint group. In the second type of holonomy group the remaining subgroup is also normal and belongs to the center of the holonomy group. The Lie group induced by the tensor $W^{\nu}_{\omega\mu\lambda} = R^{\nu}_{\omega\mu\lambda} - (1/n)R_{\omega\mu}\delta^{\nu}_{\lambda}$ has the same structural constants as $G_l(R)$ and is in one-to-one correspondence with the holonomy group of the first type. For the holonomy group of the second type this Lie group is identical with one of the two above-mentioned subgroups. The third part displays a method which yields a large class of holonomy groups with the values 0 and ± 1 for the structural constants. This method may be useful in studying all structural problems of this large class of holonomy groups. *A. Kawaguchi* (Sapporo)

12488:

Couty, R. Sur les transformations des variétés riemanniennes et kählériennes. Ann. Inst. Fourier. Grenoble 9 (1959), 147–248.

The author studies in this paper certain transformations on Riemannian and Kählerian manifolds, or more generally on manifolds with Euclidean or linear connexion. The paper is divided into two parts. In the first part, the author studies local transformations defined by the infinitesimal holonomy group, and in the second part, projective and conformal infinitesimal transformations.

Since the author makes frequent use of Lie derivatives, he recalls, at the very beginning of the paper, the definition of Lie derivatives and some important formulas.

In Chapter I of the first part, the author first states the definitions of normal coordinates and normal tensors, and then defines transformations $\mathscr{T}_R$ in the following way. In a manifold with a linear connexion, we denote by ∇_j the symbol of covariant differentiation and by $R^{i}_{\cdot jkl}$ the curvature tensor. Then the tensors

$$(R^{i}_{\cdot jj_1j_2}U^{j_1}U^{j_2})_O,\ \cdots,\ (\nabla_{j_1}\cdots\nabla_{j_r}R^{i}_{\cdot jj_1j_2}U_1^{j_1}U_2^{j_2}V_1^{j_1}\cdots V_r^{j_r})_O$$

at a fixed point O, which we denote also by

$$\overset{0}{\Omega}{}^{i}_{\cdot j},\ \overset{1}{\Omega}{}^{i}_{\cdot j},\ \cdots,\ \overset{r}{\Omega}{}^{i}_{\cdot j},$$

generate a Lie algebra of the infinitesimal holonomy group at O, where $U_1, U_2, V_1, \cdots, V_r$ are vectors in the tangent space T_O at O. To each endomorphism Ω defined by $\Omega^{i}_{\cdot j}$ corresponds a one-parameter group of automorphisms of T_O. With such an automorphism which transforms a vector Θ into a vector Θ', we associate the following local transformation. In a suitable neighborhood of O, to any point P in it corresponds a geodesic g passing through O and P. Let Θ be the vector tangent to g at O. We take, on the geodesic g' tangent to Θ' at O, a point P' which has the canonical parameter which is the same as that of P. If we denote by (x^i) the normal coordinates of P in a normal coordinate system having O as origin, the transformation is defined by the vector ξ^i whose components in a normal coordinate system having O as origin are $\xi^i = (\Omega^{i}_{\cdot j})_O x^j$. We denote by $\overset{0}{\mathscr{T}}_R, \overset{1}{\mathscr{T}}_R, \cdots, \overset{r}{\mathscr{T}}_R$ these local transformations.

In Chapter II, the author studies transformations on a Riemannian manifold. Studying the case in which transformations $\mathscr{T}_R$ are affine, he obtains: Let V be a Riemannian manifold; if the transformations $\overset{0}{\mathscr{T}}_R$ are affine, then V is a space (H). If the Ricci curvature is not degenerate [resp. if V is properly Riemannian] and if the transformations $\overset{0}{\mathscr{T}}_R, \overset{1}{\mathscr{T}}_R$ [resp. $\overset{1}{\mathscr{T}}_R, \overset{2}{\mathscr{T}}_R$] are affine, V is locally symmetric. Here a space (H) means a space in which the Cartan tensor

$$H_{mnijkl} = \nabla_m\nabla_n R_{ijkl} - \nabla_n\nabla_m R_{ijkl}$$

vanishes. The author studies also the case in which transformations $\mathscr{T}_R$ are projective or conformal and obtains a similar result. The author comes next to the case in which the transformations $\mathscr{T}_R$ preserve the volume element and proves the following. Let V be a Riemannian manifold. If the transformations $\overset{0}{\mathscr{T}}_R$ preserve the volume element, then V satisfies the infinite system of relations

$$\nabla_m\nabla_n A^{s}_{\cdot si_1\cdots i_p} - \nabla_n\nabla_m A^{s}_{\cdot si_1\cdots i_p} = 0,$$

where A's are normal tensors. If the Ricci curvature is not degenerate [resp. if V is properly Riemannian] and if the transformations $\overset{0}{\mathscr{T}}_R, \overset{1}{\mathscr{T}}_R$ [resp. $\overset{1}{\mathscr{T}}_R, \overset{2}{\mathscr{T}}_R$] preserve the volume element, the Ricci tensor of V has vanishing covariant derivative.

In Chapter III, the author studies transformations $\mathcal{T}_R$ in a complex manifold, and, for the case in which $\mathcal{T}_R$ are analytic and the manifold is Kählerian, he obtains results similar to those in Chapter II.

Chapter IV is devoted to the study of transformations $\mathcal{T}_R$ in a manifold with a Euclidean connexion.

In the second part of the paper, the author studies exclusively the infinitesimal projective and conformal transformations. In Chapter I, he studies projective and conformal transformations in a compact or complete Riemannian manifold. He considers first a conformal p-form, that is, a p-tensor $\xi_{\iota_1\iota_2\cdots\iota_p}$ satisfying

$$\nabla_j\xi_{\iota\iota_2\cdots\iota_p}+\nabla_\iota\xi_{j\iota_2\cdots\iota_p}=\frac{2}{n}\nabla^r\xi_{r\iota_2\cdots\iota_p}g_{j\iota};$$

but it seems that he does not know that for such a tensor the right-hand side is zero, because, contracting $g^{\iota\iota_2}$ to both members, we have

$$-\nabla^r\xi_{rj\iota_3\cdots\iota_p}=\frac{2}{n}\nabla^r\xi_{rj\iota_2\cdots\iota_p}.$$

He then comes to the study of affine transformations on a complete Riemannian manifold, and proves a theorem of Hano: On a complete Riemannian space an affine vector of finite length is a Killing vector. He then proves a theorem on projective vectors similar to that of Bochner: If, in a compact orientable Riemannian manifold, the Ricci curvature is negative definite, there does not exist a non-vanishing projective vector. He also proves: Let $V=W^0\times W^1\times\cdots\times W^r$ be a simply connected complete Riemannian manifold and let $\xi=\{\xi_a\}$ be a projective [resp. conformal] vector. Then the restriction of ξ_a on each W^a is a projective [resp. conformal] vector.

In Chapter II, the author considers a homogeneous Riemannian manifold G/H and shows that some of the results of Bochner, Lichnerowicz and Yano [Yano and Bochner, *Curvature and Betti numbers*, Princeton Univ. Press, Princeton, N.J., 1953; MR **15**, 989] for tensors in a compact Riemannian manifold are also valid for invariant tensors defined on G/H.

Chapter III is devoted to the study of projective and conformal transformations in an Einstein space, and the last Chapter IV to the study of those transformations in a Kählerian manifold. He proves, to show a typical example, that, on a Kählerian manifold V_{2n} $(n>1)$, any closed projective [resp. conformal] 1-form is affine [resp. homothetic].

K. Yano (Seattle, Wash.)

12489:

Cossu, Aldo. Movimenti speciali in una varietà a connessione tensoriale dotata di trasporto assoluto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 156–164.

In an affinely connected manifold admitting absolute parallelism, an infinitesimal affine motion must satisfy $\underset{v}{\pounds}\,\lambda_b{}^h=C_b{}^a\lambda_a{}^h$, where $\underset{v}{\pounds}$ denotes the Lie derivative with respect to v, $\lambda_a{}^h$ are absolutely parallel vectors and the C's are constant. When the C's are all zero, the affine motion is called special [K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957; MR **19**, 576]. In the paper under review, the author studies analogous transformations in a space with tensorial connexion admitting an absolute parallelism [E. Bompiani, same Atti (8) **1** (1946), 478–485; MR **8**, 404].

K. Yano (Seattle, Wash.)

12490:

Levin, Yu. I. Some spaces with affine connection admitting motions. Dokl. Akad. Nauk SSSR **137** (1961), 1295–1298 (Russian); translated as Soviet Math. Dokl. **2**, 464–467.

One of the theorems proved is the following. The most general A_n without torsion, admitting a simply transitive group of affine collineations with operators $X_\alpha=\xi^i\partial/\partial x^i$ $(i,j,k,\alpha,\beta,\gamma=1,2,\cdots,n)$ has the form

$$\Gamma_{ij}{}^k=\tfrac{1}{2}\overset{(\alpha)}{\lambda}{}^k\left(\frac{\partial\overset{(\alpha)}{\lambda}_i}{\partial x^j}+\frac{\partial\overset{(\alpha)}{\lambda}_j}{\partial x^i}\right)+k_{\beta\gamma}{}^\alpha\overset{(\beta)}{\lambda}{}^k\overset{(\gamma)}{\lambda}_i\overset{(\gamma)}{\lambda}_j,$$

where the $\overset{(\alpha)}{\lambda}{}^i$ are the coefficients of the group reciprocal to the group X_α, so that $\overset{(\alpha)}{\lambda}_i\overset{(\alpha)}{\lambda}{}^j=\delta_i{}^j$; $k_{\beta\gamma}{}^\alpha=k_{\gamma\beta}{}^\alpha$ are arbitrary constants. The affine collineations of such a space are then and only then displacements when $k_{\beta\gamma}{}^\alpha=\delta^\alpha_{(\beta}e_{\gamma)}$. Then the investigation continues with tensors admitting simply transitive abelian groups of automorphisms.

D. J. Struik (Cambridge, Mass.)

12491:

Poliščuk, E. M. Isogeneity and Riemannian metrics. Ukrain. Mat. Ž. **11** (1959), 66–82. (Russian. English summary)

This paper contains additional results in the direction of the author's earlier paper [Vestnik Leningrad. Univ. **12** (1957), no. 13, 27–49; MR **21** #2257].

Author's summary: "The author considers couples of complex (double) functionals

$$F[\Gamma_p]=\int_{\Gamma_p}dF,\quad \phi[\Gamma_p]=\int_{\Gamma_p}d\phi$$

of bounding cycles Γ_p in a compact orientable n-dimensional space E_n of variables $x^1, x^2, \cdots, x^n$.

"Here

$$dF=dF_1+idF_2,\quad d\phi=d\phi_1+id\phi_2$$

$$(dF=dF_1+edF_2,\quad d\phi=d\phi_1+ed\phi_2)$$

are complex (double) differential skew-symmetric closed $p+1$ forms of variables $x^1, x^2, \cdots, x^n$; e is an element of Clifford's algebra: $e^2=1$.

"If $d\phi=fdF$, where f is a complex (double) scalar field in E_n, then F and ϕ are called isogeneous and f is a 'derivative' $d\phi/dF$.

"A geometrical interpretation of the modulus ρ and the argument φ of the derivative $d\phi/dF$ is given in which ρ and φ are treated as the homothety coefficient and the rotation angle of fields of couples of 'conjugates' orthogonal vectors in some Riemannian spaces V_{2N}, $V_{2N}{}^*$ $(N=C_n{}^p)$.

"Differential invariants of spaces V_{2N}, $V_{2N}{}^*$ are established, as well as of some of their subspaces, analogous to those found by Hodge in his theory of harmonic integrals."

M. M. Day (Urbana, Ill.)

12492:

Pogorelov, A. V. The regularity of convex surfaces with a regular metric in Lobačevskiĭ space. Dokl. Akad. Nauk

SSSR **137** (1961), 35–38 (Russian); translated as Soviet Math. Dokl. **2**, 235–237.

The author's results on differentiability properties of convex surfaces with positive curvature in the euclidean and elliptic spaces can be extended to hyperbolic space, but the result obtained in this way is not natural, because the proper analogue to positive curvature is in this case that the curvature of the surface should exceed the curvature of the space. The present article fills this gap: If a (not necessarily closed) convex surface in a hyperbolic space of curvature κ^2 has a regular metric with E, F, G of class C^k, $k \geqq 5$, and has curvature greater than $-\kappa^2$ then the surface is at least of class C^{k-1}.

The principal tool is here, as in the euclidean case, an a priori estimate for the normal curvatures of regular convex caps. *H. Busemann* (Los Angeles, Calif.)

12493:

Hsü, Chin-shui. A remark on the characterization of homothetic transformation and inversion. Proc. Amer. Math. Soc. **11** (1960), 685–686.

Two theorems of the author's previous paper [same Proc. **10** (1959), 324–328; MR **21** #6601] are generalized as follows. Consider two closed orientable surfaces S and $\bar{S}$ in E^3 and suppose the existence of a diffeomorphism h: $S \to \bar{S}$ such that $\bar{X} = kX$ ($k \neq 0$). X, $\bar{X}$ are position vectors of S, $\bar{S}$ relative to O; H, $\bar{H}$ the mean curvatures; N, $\bar{N}$ the unit inward normals, and $p = -X \cdot N$. Suppose moreover that S, $\bar{S}$ contain no pieces of cones with vertex O. Then if $\int_S (H \mp k\bar{H}) p\, dA = 0$, k is constant, and h is a homothetic transformation with center O. Also if

$$\int_S [H \pm k(\bar{H} + 2\bar{X} \cdot \bar{N}/\bar{X} \cdot \bar{X})]\, dA = 0$$

and O is not a point of S or $\bar{S}$, then h is an inversion with center O. *C. B. Allendoerfer* (Seattle, Wash.)

12494:

Hsü, Chin-shui. Generalization of Cohn-Vossen's theorem. Proc. Amer. Math. Soc. **11** (1960), 845–846.

Let S, $\bar{S}$ be two orientable, closed, convex surfaces of class C^3 in E^3, and consider a diffeomorphism h: $S \to \bar{S}$ which preserves KI, where K is the Gaussian curvature and I the first fundamental form. Then h is a similarity. The hypotheses can be weakened to include the analytic T surfaces of Alexandrov.

C. B. Allendoerfer (Seattle, Wash.)

12495:

Bublik, B. A. On the existence of non-rigid closed surfaces. Dokl. Akad. Nauk SSSR **131** (1960), 725–727 (Russian); translated as Soviet Math. Dokl. **1**, 310–312.

The author constructs an example of a closed, twice continuously differentiable, non-rigid surface admitting two linearly independent bending fields. An example of a closed non-rigid surface with one such field was given by Cohn-Vossen in 1929. *A. V. Pogorelov* (Kharkov)

12496:

Rešetnyak, Yu. G. On the theory of spaces with curvature no greater than K. Mat. Sb. (N.S.) **52 (94)** (1960), 789–798. (Russian)

A. D. Aleksandrov [Trudy Mat. Inst. Steklov. **38** (1951), 5–23; MR **14**, 198] has defined a space with curvature no greater than K to be a region R_K in a metric space with the following properties: (1) Any two points in R_K can be joined by a geodesic; (2) any triangle in R_K with perimeter less than $2\pi/k$, where $K = k^2 > 0$, has negative K-excess.

The author proves a number of theorems for such spaces, in particular, the following. If two surfaces R' and R'' of curvature no greater than K are pasted together by identifying those points of two mutually isometric compact convex sets $V' \subset R'$, $V'' \subset R''$ corresponding to one another under the isometry, the resulting space is of curvature no greater than K.

A. V. Pogorelov (Kharkov)

12497:

Pogorelov, A. V. The rigidity of closed surfaces non-homeomorphic to a sphere in Riemannian space. Dokl. Akad. Nauk SSSR **138** (1961), 51–52 (Russian); translated as Soviet Math. Dokl. **2**, 537–539.

The author, in the paper *Nekotorye voprosy geometrii v celom v rimanovom prostranstve* [Izdat. Har'kov. Univ., Kharkov, 1957; MR **20** #4303], proved the rigidity of closed surfaces in Riemannian space, when they are homeomorphic to a sphere. Here he proves an analogous theorem for a closed surface homeomorphic to a torus with positive external curvature in Riemannian space. Such a surface, fixed at one point, is rigid, and, in particular, does not admit of a continuous bending. The theorem appears to be true for any genus p, and for $p > 1$ without any conditions concerning the fixation.

D. J. Struik (Cambridge, Mass.)

12498:

Ryabčikova, I. Ya. Infinitesimal bending of surfaces in Lobačevskiĭ space. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 173–176. (Russian)

A l'aide de la représentation géodésique de l'espace de Lobatchevsky sur l'intérieure d'une sphère euclidienne, l'auteur démontre que la déformation infiniment petite de la surface F en l'espace de Lobatchevsky entraîne une déformation infiniment petite de la représentation géodésique de F et vice versa. Il en suit les propositions: (a) Une surface régulière convexe fermée sans domaines planes est rigide dans l'espace de Lobatchevsky. (b) Pour les surfaces du seconde ordre toutes les déformations infiniment petites peuvent être déterminées en termes finis. (c) Si les déformations infiniment petites de la surface F sont connues, les déformations infiniment petites de la surface, obtenue de F par une transformation projective, peuvent être déterminées également.

S. P. Finikov (Moscow)

12499:

Santaló, L. A. Two applications of the integral geometry in affine and projective spaces. Publ. Math. Debrecen **7** (1960), 226–237.

The author gives first the density, invariant with respect to the unimodular affine group, for sets of pairs $\mathscr{P}$ of parallel hyperplanes. Let $\mathscr{P}$ be $\sum l^i x_i = m$, $\sum l^i x_i = m+1$. The density takes the form $d\mathscr{P} = dl^1 \wedge dl^2 \wedge \cdots dl^n \wedge dm$, and its geometric interpretation is expressed by the form

$$d\mathscr{P} = \frac{d\sigma \wedge dp_1 \wedge dp_2}{|p_2 - p_1|^{n+2}},$$

where p_1, p_2 are the distances from the origin to two hyperplanes and σ is the element of area on the unit sphere. The measure of the set of pairs of parallel hyperplanes which contain a given convex body K is related with the volume of K by two inequalities, where the upper bound is attained if and only if K is an ellipsoid. Next the author gives three forms of the density for sets of hyperquadrics invariant with respect to the projective group, the third of which generalizes the formula given for conics on the plane by Stoka [Acad. R. P. Romîne Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 903–937; MR **18**, 63]. Let the coefficients of a hyperquadric be $Q=(q_{hl})$, where $|Q|=\pm 1$. The first form of the density is given by

$$dC = \frac{1}{n+1}\sum_{h,l=0,\ h\le l}^{n}(-1)^{\nu(h,l)}q_{hl}\,dq_{00}\wedge dq_{01}\wedge\cdots \wedge \widehat{dq_{hl}}\wedge\cdots\wedge dq_{nn}$$

$$(\nu(h,l)=\tfrac{1}{2}(2n+1-h)h+l+1).$$

H. Hombu (Tokyo)

12500:

Soós, Gy. Über einfache Finslersche Räume. Publ. Math. Debrecen **7** (1960), 364–373.

A special class of Finsler spaces is considered, namely, such spaces for which the conditions $A_{ijk|m}=0$ and $A_mR_0{}^m{}_{kh}=0$ hold identically everywhere. (The notation is that of E. Cartan: *Les espaces de Finsler*, Hermann, Paris, 1934.) The author calls these "simple" (einfach) Finsler spaces. It is shown that the first condition taken together with the condition $R_0{}^j{}_{kh}=0$ also ensures that the space is simple and that under these conditions it is actually Minkowskian. Furthermore, a simple Finsler space of scalar curvature is either Minkowskian or a Riemannian space of constant curvature. A simple space with vanishing projective curvature is a space of scalar curvature. Two Finsler spaces F_n and $\bar{F}_n$ are said to be mapped geodesically onto each other, if there exists a relation of the form $\bar{G}^i=G^i+Q(x^k,\ \dot{x}^k)\dot{x}^i$ between the functions $\bar{G}^i$ and G^i of $\bar{F}_n$ and F_n respectively, Q being a scalar function homogeneous of the first degree in the $\dot{x}^k$. Simple relations between the connection parameters and the curvature tensors of $\bar{F}_n$ and F_n are obtained, assuming that the latter are simple. Finally it is shown that a simple F_n $(n\neq 2)$ is projectively flat if under an infinitesimal projectivity the covariant derivative of the projective curvature tensor remains invariant.

H. Rund (Durban)

12501:

Busemann, H.; Straus, E. G. Area and normality. Pacific J. Math. **10** (1960), 35–72.

An "a-flat" is an a-dimensional linear subspace of affine n-space A^n. A, B, $\cdots$ denote a-flats, b-flats, etc. Let $0\le d<\min(a,\ b)$, $q=a+b-d\le n$. For any $a+1$ points $x_0, x_1, \cdots, x_a$ of A^n put

$$[x_0, x_1, \cdots, x_a] = \{x \mid x = x_0+\sum_1^a \theta_i(x_i-x_0);\ 0\le\theta_i\le 1\}.$$

"An a-area α assigns to every Borel set M in an a-flat a measure $\alpha(M)$ which is invariant under the translations of A^n" such that $\alpha([x_0, \cdots, x_a])$ is a continuous function of the simple a-vector $\mathfrak{A}=(x_1-x_0)\wedge(x_2-x_0)\wedge\cdots\wedge(x_a-x_0)$, say

$$\alpha([x_0, \cdots, x_a]) = F(x_1-x_0, \cdots, x_a-x_0) = a!\phi(\mathfrak{A})$$

where $\phi(\mathfrak{A})>0$ if $\mathfrak{A}\neq 0$ and $\phi(\lambda\mathfrak{A})=|\lambda|\cdot\phi(\mathfrak{A})$ for all real λ.

Suppose $A\cap B=D$, $A+B=Q$. Let C be a $(b-d)$-flat in B which intersects D in exactly one point and let A_0 be any a-flat through D in Q. If M is the projection parallel to C onto A of the Borel set $M_0\subset A_0$ and if $0<\alpha(M_0)<\infty$, then $\alpha(M)/\alpha(M_0)$ is independent of the choice of C and M_0. If $\alpha(M)/\alpha(M_0)\le 1$ for all A_0, A is "totally normal" to B (B is "totally transversal" to A) at D in Q. A is "normal" to B (B is "transversal" to A) at D in Q if A is totally normal at D in Q to every $(d+1)$-flat in B through D.

The a-area α is called convex, strictly convex, or differentiable, if the curve $F(\lambda_1y_1+\lambda_2y_2, x_2, \cdots, x_a)=1$ has these properties in the $(\lambda_1,\ \lambda_2)$-plane whenever $y_1\wedge y_2\wedge x_2\wedge\cdots\wedge x_a\neq 0$. We quote at least the following results: Let $d=\min(a,\ b)-1$, $D\subset A\subset Q$. Then there exists a b-flat transversal to A at D in Q if and only if α is convex. It is unique if α is differentiable. The normal to B at D in Q always exists. It is unique for all $D\subset B\subset Q$ if and only if α is strictly convex.

Two convex a-areas which determine the same normality relations are proportional. The convex a-area α and the convex b-area β may be called d-dual if α-normality and β-transversality of a-flats to b-flats at d-flats are equivalent. If $F^2(x_1, x_2, \cdots, x_a)$ is a quadratic form in the components of each x_i, α is called quadratic. If $a<n$, $b<n$ and if α and β are d-dual, then they are both quadratic unless $a+b=n$, $d=0$. In particular, a "symmetric" [=self-dual] a-area is quadratic unless $a=n/2$, $d=0$. Every convex $(n-1)$-area is 0-dual to some convex 1-area [the Minkowski metric].

The a-area α is "euclidean" in $A^m\subset A^n$ if there is a euclidean metric in A^m such that the α-area of every $[x_0, x_1, \cdots, x_a]\subset A^m$ is equal to its a-dimensional euclidean volume. We have: α is quadratic $\leftrightarrow$ α is euclidean in every $(a+1)$-flat $\leftrightarrow$ α is symmetric in every $(a+1)$-flat. Thus quadratic $(n-1)$-areas are euclidean. A quadratic a-area is euclidean if it is euclidean in every $(a+2)$-flat.

The a-area α is "totally convex" if there exists an $(n-a)$-flat totally transversal to any given a-flat at a point; α is "extendably convex" if $\phi(\mathfrak{A})$ can be extended to a convex function defined for all a-vectors. Total convexity implies extendable convexity, but even a quadratic extendably convex a-area need not be totally convex. Convexity does not imply extendable convexity. If α is extendably convex, "the a-flats minimize area in the sense that the α-area of the union of all but one face of a closed a-dimensional polyhedron is not less than the area of that face. . . . On the other hand for $1<a<n-1$ the a-flats need not minimize α-area when α is merely convex."

Let $1\le a\le n-1$. "The a-dimensional Minkowski area in an n-dimensional Minkowski space with distance $F(x-y)$ is the area of the above type for which . . . the set $\{x \mid F(x-x_0)\le 1;\ x, x_0\in A\}$ has the euclidean volume $\pi^{a/2}\Gamma(a/2+1)$" for every A, x_0. This area conversely determines $F(x-y)$. It is convex. It is strictly convex [differentiable] if the unit sphere is strictly convex [differentiable]. If it is quadratic, the space is euclidean. If a Minkowski a-area is d-dual to a Minkowski b-area, both Minkowski metrics of A^n are therefore euclidean ($a<n, b<n$ but not $a+b=n, d=0$). *P. Scherk* (Toronto)

GENERAL TOPOLOGY, POINT SET THEORY

See also 12196, 12416, 12538.

12502:

Kowalsky, H.-J. ★Topologische Räume. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Bd. 26. Birkhäuser Verlag, Basel-Stuttgart, 1961. 271 pp. sFr. 40.00.

This book presents an introduction to general topology; that is, it treats the general ideas, such as convergence and functions, which are often studied by using a topology. Consequently, the major emphasis is on very weak topological properties such as the separation axioms and covering properties. However, some special topics are included. For example, the continuous images of the unit interval are characterized, which means that some complicated proofs concerning connectedness are illustrated.

In Chapters one and two, the concept of a topological space is introduced. Chapter one contains the necessary intuitive set theory and introduces filters, which are to be used in convergence and to define the topology. In Chapter two, topological spaces are defined in terms of neighborhood systems of their points. The separation axioms and countability axioms are introduced here, and metric spaces are defined.

Chapters three, four and five contain the introductory material on covering properties, continuous functions, and imbeddings. Compact and paracompact spaces are introduced in Chapter three. Continuous functions, product spaces, quotient spaces, and completely regular spaces are discussed in Chapter four. Chapter five combines the ideas of the preceding two chapters in the theorems on imbeddings and compactifications.

Chapters six and seven treat uniform spaces and their connection with topological groups, rings, fields, etc. In Chapter six, metric spaces and metrization theorems are studied, and then the more general uniform structures are introduced in terms of neighborhoods of the diagonal. Chapter seven presents some of the consequences of having an algebraic structure as well as a topology. For example, associated uniform structures are studied, and the Stone-Weierstrass theorem is proved.

It should be noted that through the book runs a string of useful results on ordered spaces and connected spaces, which might not be expected from the above description.

Some topics which, in the reviewer's opinion, could have been and should have been treated, or treated more thoroughly, are: the Baire category theorem and its consequences, partitions of unity, function spaces, the first principles of dimension theory. Perhaps problems could be constructed to illustrate these ideas.

H. H. Corson (Seattle, Wash.)

12503:

Бурбаки, Н. [Bourbaki, N.]. ★Общая топология: Числа и связанные с ними группы и пространства [General topology: Numbers and the groups and spaces related to them]. Translated from the French by C. N. Kračkovskiĭ; edited by D. A. Raĭkov. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 247 pp. 9 r.

A translation into Russian of part of the second edition of *Livre III. Topologie générale* of Bourbaki [Actualités Sci. Ind., No. 916, 1029; Hermann, Paris, 1942, 1947; MR **5**, 102; **9**, 261], containing the following numbering of chapters: Chap. IV, Real numbers; Chap. V, One-parameter groups; Chap. VI, Number spaces and projective spaces; Chap. VII, Additive groups of R^n; Chap. VIII, Complex numbers.

12504:

Górski, J. Les suites de points extrémaux liés aux ensembles dans l'espace à 3 dimensions. Ann. Polon. Math. **4** (1957), 14–20.

12505:

Górski, J.; Siciak, J. Certains théorèmes concernant la répartition des points extrémaux dans les ensembles plans. Ann. Polon. Math. **4** (1957), 21–29.

12506:

★Séminaire de topologie et de géométrie différentielle dirigé par Charles Ehresmann. 1re année: 1957/58. Faculté des Sciences de Paris. Secrétariat mathématique, Paris, 1959. 155 pp. (mimeographed)

See the individual reviews: #12507, 12508, 12359, 12464, 12483, 12523, 12524, 12358a.

12507:

Papert, Dona; Papert, Seymour. Sur les treillis des ouverts et les paratopologies. Séminaire C. Ehresmann, 1957/58, exp. no. 1, 9 pp. Faculté des Sciences de Paris, 1959.

A paratopology is a complete relatively pseudocomplemented lattice. In the category of paratopologies the mappings are the homomorphisms which preserve infinite joins. There is a natural contravariant functor from topological spaces to paratopologies, which for Hausdorff spaces is an anti-isomorphism.

The authors first study distributivity and the way the category of all topological spaces maps into the category of paratopologies. Then they state a number of definitions and theorems generalized from topology, concluding with a Tychonoff product theorem and some remarks about different notions of product and examples.

{Propositions 3 and 5 cannot be correct as printed. The first suffers from the conditions $G \in P(P(L))$, $F \in P(P(L))$, $G \in F$ in the definition of F-distributivity; the second would make every finite lattice a topology. There is further trouble somewhere in the definitions, for Theorem iii makes every connected paracompact paratopology locally compact and σ-compact. Further, the authors' notion of product is what is usually called a direct sum.}

J. Isbell (Lafayette, Ind.)

12508:

Benabou, Jean. Treillis locaux et paratopologies. Séminaire C. Ehresmann, 1957/58, exp. no. 2, 27 pp. Faculté des Sciences de Paris, 1959.

This paper is mainly concerned with the same category of objects as in #12507, here called local lattices. A paratopology is a family of elements of a local lattice closed under finite intersections and arbitrary unions, and the first part of the paper is a collection of elementary observations on these, not used in the sequel. The second part is a study of embeddings of semilattices (under $\wedge$)

in complete lattices, especially in local lattices. With a suitable definition of mappings (agreeing with #12507 for local lattices) there is an extremal embedding of each semilattice P in a local lattice Φ_0, characterized by the property that each mapping of P into a local lattice has a unique extension over Φ_0.

The concluding section establishes essential portions of paratopological generalizations of the theorems of Tychonoff and Gleason. First, every family of local lattices has a direct sum. (Combined with #12507 this yields a whole Tychonoff theorem for paratopologies in the sense of #12507.) Second, every local lattice can be embedded in a complete Boolean algebra, the embedding being a mapping in the present sense. *J. Isbell* (Lafayette, Ind.)

12509:

Gleason, Andrew M. Projective topological spaces. Illinois J. Math. **2** (1958), 482–489.

In the usual terminology of homological algebra the author shows that in the category of all compact Hausdorff spaces and all continuous maps the projective objects (i.e., spaces) are precisely the extremally disconnected spaces (i.e., those spaces where the closures of open sets are open). Further, every object in this category is the image of a projective object, namely, the compact space X is a continuous image of the Stone space of the complete Boolean algebra of all regular closed subsets of X (a set is regular closed, or a closed domain, if it is equal to the closure of its interior). This specific projective resolution is shown to be unique in a suitable sense. The method of proof reminds the reviewer of the Bourbaki use of ultrafilters in characterizing compact spaces. Also Lemma 3.1 (the regular closed sets form a complete Boolean algebra) is not new [see, e.g., J. C. C. McKinsey and A. Tarski, Ann. of Math. (2) **47** (1946), 122–161; MR **7**, 359]. The analogous results are stated for the category of locally compact spaces (warning: do not use all continuous maps) and then by duality for commutative C^* algebras. The dual result for the category of compact totally disconnected spaces (i.e., for Boolean algebras) had been obtained earlier by R. Sikorski, as the author points out.

Dana Scott (Chicago, Ill.)

12510:

Isiwata, Takesi. Characterizations of spaces with dual spaces. Proc. Japan Acad. **36** (1960), 200–204.

Let $X^*=\beta(X)-X$, where βX is the Stone-Čech compactification of X. Then the author defines a space with a dual space as an X such that there is a homeomorphism of $\beta(X^*)$ onto $\beta(X)$ which leaves each point of X^* fixed. The author gives conditions which are equivalent to, or imply that, X is a space with a dual space. Several examples are given. *H. H. Corson* (Seattle, Wash.)

12511:

Sklyarenko, E. G. Perfect bicompact extensions. Dokl. Akad. Nauk SSSR **137** (1961), 39–41 (Russian); translated as Soviet Math. Dokl. **2**, 238–240.

The perfect compactifications BX with which this paper is concerned turn out to be those for which the natural mapping $\beta X \to BX$ is monotone. They are characterized in several ways; e.g., $BX \setminus X$ is not locally separating anywhere in BX. If there is a perfect compactification with punctiform complement, it is unique. (This generalizes a theorem of R. Duda for compact metric spaces [Nederl. Akad. Wetensch. Proc. Ser. A **63** (1960), 132–136; MR **22** #969].) In the rim-compact case, this is the Freudenthal compactification, and more details are given concerning the partially ordered set of all compactifications. *J. R. Isbell* (Seattle, Wash.)

12512:

Ponomarev, V. I. Properties of topological spaces preserved under many-valued continuous mappings. Mat. Sb. (N.S.) **51 (93)** (1960), 515–536. (Russian)

Here "mapping" means "multiple-valued mapping". It is shown that perfect mappings (a suitable generalization of proper mappings) preserve paracompactness, local bicompactness, and in fact all the "fitting" properties of Henriksen and Isbell [Duke Math. J. **25** (1957), 83–106; MR **20** #2689]. The author had proved this for two of the properties before learning of the work of Henriksen and Isbell.

The properties of light compactness and absolute H-closedness are preserved forward by certain mappings, called strongly continuous Y-bicompact mappings. Some results are also formulated on rim-compactness and on local connectedness. *J. R. Isbell* (Seattle, Wash.)

12513:

Ponomarev, V. I. Extension of many-valued mappings of topological spaces to their compactifications. Mat. Sb. (N.S.) **52 (94)** (1960), 847–862. (Russian)

The author seeks extensions of mappings [of the sort considered in #12512] over Wallman compactifications ωX and over proximity compactifications υX. A closed continuous Y-bicompact mapping from X to Y has a unique smallest closed continuous extension on ωX to ωY. For proximity spaces, strongly uniform mappings are defined, and it turns out that strongly uniform Y-bicompact mappings from X to Y are precisely those which have strongly continuous extensions on υX to υY.

J. R. Isbell (Seattle, Wash.)

12514:

Okuyama, Akihiro. On multi-valued monotone closed mappings. Proc. Japan Acad. **36** (1960), 106–110.

The author characterizes the multi-valued inverses of single-valued closed continuous mappings in T_1 spaces. He also proves two theorems on preservation of properties by multi-valued mappings which, compared with Ponomarev's theorems [cf. #12513], have stronger hypotheses on the spaces and incomparable hypotheses on the mappings. *J. R. Isbell* (Seattle, Wash.)

12515:

Doyle, P. H.; Hocking, J. G. A characterization of Euclidean n-space. Michigan Math. J. **7** (1960), 199–200.

Let M be an m-manifold, and suppose there is a point p in M such that, for each neighborhood U of p, there exists a homeomorphism h of M onto itself such that $h(M-U) \subset U$. Then M is an m-sphere. The authors' proof is based on a theorem of the reviewer [Bull. Amer. Math. Soc. **66** (1960), 74–76; MR **22** #8470b]. A similar theorem gives a characterization of E^m.

Morton Brown (Princeton, N.J.)

12516:

Jaworowski, J. W. An application of the method of complex functions to a proof of Kakutani's theorem. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 169–171. (Russian summary, unbound insert)

The proof is essentially the same as Kakutani's.

D. G. Bourgin (Urbana, Ill.)

ALGEBRAIC TOPOLOGY

See also 12131, 12132, 12136, 12416, 12417, 12530, 12531, 12533.

12517a:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. X. Über Heawood'sche Kantengleichungen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **8** (1958/59), 1067–1072.

12517b:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. XI. Elementare Eigenschaften von Vierkantnetzen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **9** (1960/61), 103–108.

12517c:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. XII. Heawoodsche Eckengleichungen für halbgerade Dreikantnetze auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **9** (1960/61), 109–114.

For basic terminology and reviews of the first nine notes of this series the reader is referred to same Z. **5** (1955/56), 839–844; **6** (1956/57), 697–704, 785–788, 789–798; **7** (1958), 353–358, 447–456; **8** (1958/59), 109–120, 337–344, 747–754 [MR **22** #2987a–d, 7113a–e]. The present review covers the (presumed) final three papers of the series.

X. Concerning Heawood's edge equation. Suppose N an arbitrary net of an arbitrary closed orientable or non-orientable surface T. Then it is possible to define a Heawood signature function $\sigma(k) = \pm 1$, k an edge of T, in such a way that (Theorem 1) if the number of edges of N is even, then for each even region, if any, $\sum \sigma = 0$ and for each odd region, if any, $\sum \sigma = \pm 1$. In addition, by arbitrarily pairing the odd regions in advance, the assignment can be made so that $\sum \sigma = +1$ for one member of each pair and $\sum \sigma = -1$ for the other member of the pair. Three more theorems follow which cover the remaining possibilities of edge and region distributions [e.g., if the number of edges of N is odd but N has odd regions only]. In general the note is concerned with certain Heawood edge equations for arbitrary nets, whose solutions produce a possible "uniform" distribution of a two-valued edge signature analogous to the results in the theorem of IX.

XI. Elementary characterization of four-edge nets: The paper contains some remarks concerning four nets on a sphere K_0 analogous to those in I. In addition, the following theorem is proved. If each edge is connected, each four-edge net on an arbitrary closed carrier T which possesses exactly n generators can be transformed by appropriate "decomposition" of its vertices, into an even three-edge net $N_3(2, 0)$ in at least 2^{n-1} different ways.

With a more or less arbitrary net N on the sphere can be associated in XII a semi-even three-net denoted hg N_3 and elements of a certain "regular" subclass of these are called hg N_3*'s. With respect to this class the following signature theorem is proved:

For each hg N_3* (of the sphere) there is a Heawood vertex signature function $\sigma(E) = \pm 1$ such that, for each region of hg N_3*, $\sum \sigma(\Gamma)$ [the sum of the signatures for the vertices of Γ] is 0 or ± 3 according as Γ is an even or odd region. Furthermore the assignment may be made to satisfy certain other side conditions. A number of observations are made concerning the possibility and impossibility of solving various Heawood vertex equations for semi even three nets on the sphere.

L. M. Kelly (E. Lansing, Mich.)

12518:

Dauker, K. H. [Dowker, C. H.] The Kolmogorov-Aleksandrov duality theorem. Mat. Sb. (N.S.) **50 (92)** (1960), 247–255. (Russian)

This paper is concerned with cohomology relative to a family of supports. The author uses the following notions. (1) If Φ is a family of supports for X, $A \subseteq X$, then Φ_A is the collection of sets $F \cap A$, for $F \in \Phi$. (2) If $G \subseteq X$ is open, then $\Phi(G)$ is the collection of all $F \in \Phi$ which are contained in G. He then defines the concept of Φ-good position of a closed set E, and proves the following two theorems.

Theorem: Let Φ be a family of supports for X. Let E be a closed subset of X, which is in Φ-good position. Set $G = X - E$. Then the natural homomorphism $\mu: H^q_{\Phi(G)}(G) \to H_\Phi{}^q(X, E)$, q an integer ≥ 0, is an isomorphism. Theorem: With Φ and X as above, let $A \subseteq X$. Let G be open in X such that $G \cup A = X$ and $E = X - G$ is in Φ-good position in X and in Φ_A-good position in A. Then the natural homomorphism $H^q_{\Phi(G)}(G, G \cap A) \to H_\Phi{}^q(X, A)$, $q \geq 0$, is an isomorphism.

Both of the above theorems give rise to exact sequences in a direct manner. Finally, the author proves the theorem: Let $f: (X, A) \to (Y, B)$ be a closed map such that $f^{-1}B = A$. Let H be open in Y such that $H \cup B = Y$. Set $G = f^{-1}(H)$. Suppose that $f|G$ is a homeomorphism, and that ∂G is paracompact and has a collectionwise normal closed neighborhood N in the closure of G. Then the induced homomorphism $f^*: H^q(Y, B) \to H^q(X, A)$ is an isomorphism.

D. W. Kahn (New Haven, Conn.)

12519a:

Nakaoka, Minoru. Cohomology mod p of the p-fold symmetric products of spheres. J. Math. Soc. Japan **9** (1957), 417–427.

12519b:

Nakaoka, Minoru. Cohomology mod p of symmetric products of spheres. J. Inst. Polytech. Osaka City Univ. Ser. A **9** (1958), 1–18.

12519c:

Nakaoka, Minoru. Cohomology mod p of symmetric products of spheres. II. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 67–89.

These papers completely determine the mod p cohomology (p a prime) of symmetric products of the

n-sphere S^n (=orbit space of $S^n \times \cdots \times S^n$ under the action of the symmetric group), including cup-products and Steenrod powers.

Following Serre-Cartan the author considers all sequences $I=(i_1, i_2, \cdots)$ of non-negative integers i_k such that (1) $i_k \equiv 0$ or 1 mod $2(p-1)$, $i_k \neq 1$, (2) $i_k \geq p i_{k+1}$ (hence almost all $i_k=0$), (3) $pi_1 < (p-1)D_n(I)$ where $D_n(I)=n+i_1+i_2+\cdots$. Let $U(p, n)$ denote the free commutative graded Z_p-algebra generated by these sequences, the degree of a generator I being $D_n(I)$ (if $p=2$, U is polynomial; if $p>2$ it is a tensor product of a polynomial and an exterior algebra). For every space X and cohomology class $y \in H^n(X)=H^n(X_0, Z_p)$, define an algebra homomorphism T_y: $U(p, n) \to H^*(X)$ by $T_y(I)=\mathrm{St}^{i_1} \circ \mathrm{St}^{i_2} \circ \cdots (y)$ where St^i is that Steenrod power which raises dimension by i (i.e., of the form Sq^i or $\mathscr{P}^s$ or $\delta^* \mathscr{P}^s$). If $\mathfrak{S}_m(S^n)$ now denotes the m-fold symmetric product of S^n, and $u(m) \in H^n(\mathfrak{S}_m(S^n))=Z_p$, a generator, then the main theorem states: $T_{u(m)}$: $U(p, n) \to H^*(\mathfrak{S}_m(S^n))$ is an epimorphism whose kernel is generated by all elements of p-length $\geq m$. The p-length is another gradation of $U(p, n)$, the q-length of a generator $I=(i_1, i_2, \cdots)$ being p^i if $i_l \neq 0$, $i_{l+1}=0$. This result not only describes $H^*(\mathfrak{S}_m(S^n))$ as a ring but also as a module over the Steenrod algebra: one has only to recall Cartan's product formula and Adem's relations.

The distribution over the three papers is as follows: The first treats the case $m=p$ (and does not use $\mathfrak{S}_\infty(S^n)$); in the second, m is arbitrary but only the stable cohomology $H^{n+q}(\mathfrak{S}_m(S^n))$, $q<n$, is computed. The third contains the general result. The author notes that his method can be used to compute $H^*(\mathfrak{S}_m(K))$ for more general complexes K. This seems clear for Moore-spaces K (only one homology group $\neq 0$) or wedges of those, and hence for the additive structure of $H^*(\mathfrak{S}_m(K))$ in general; the ring structure, however, or the operations St may cause new difficulties.

A. Dold (New York)

12520:

Browder, William. Some additive cohomology operations which are not suspensions. Proc. Cambridge Philos. Soc. **57** (1961), 50–54.

Let $x \in H^{2q}(K(Z_p, 2q-1); Z)$ be a generator, where p is an odd prime. Moore [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 409–411; MR **22** #1893] showed that $(x)^p$ defines an additive cohomology operation which is not a suspension. In this paper, the author generalizes this result to the following theorem. Let $x \in H^{2q}(K(\pi, n); Z)$ define an additive cohomology operation. Let x be of order p, an odd prime. Let $n>1$ and π be finitely generated. Let $j_*(x) \neq 0$, where j: $Z \to Z_p$ is reduction mod p. Then $(x)^p$ defines an additive cohomology operation which is not a suspension. To prove this theorem, the author assumes that $(x)^p$ is the suspension of z and studies the spectral sequence of the path space of Y, where Y is a space with two homotopy groups and k-invariant equal to z.

F. P. Peterson (Oxford)

12521:

Umeda, Yoshio. A remark on a theorem of J. P. Serre. Proc. Japan Acad. **35** (1959), 563–566.

Verf. beweist einen Satz, der in einem Spezialfall ($p=2$) von J.-P. Serre [Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643] stammt. Der Beweis verläuft ähnlich wie bei Serre. Es werden die Ergebnisse von H. Cartan über die Homologiegruppen $H_*(\pi, n; Z_p)$ herangezogen [Séminaire Henri Cartan de l'École Normale Supérieure, 1954/1955, Sécretariat mathématique, Paris, 1955; MR **19**, 438]. "Theorem: Let p be an odd prime, and let X be an arcwise and simply connected topological space satisfying: (i) $H_i(X, Z)$ is finitely generated for all $i>0$; (ii) $H_i(X, Z_p)=0$ for all sufficiently large i; (iii) $H_i(X, Z_p) \neq 0$ for some $i>0$. Then there exist infinitely many values of i such that $\pi_i(X)$ has a subgroup isomorphic to Z or Z_p."

F. Hirzebruch (Bonn)

12522:

★Расслоенные пространства и их приложения [Fiber spaces and their applications]. Collection of translations, edited by V. G. Boltyanskiĭ, E. B. Dynkin, M. M. Postnikov. Izdat. Inostr. Lit., Moscow, 1958. 460 pp. 30.15 r.

In order to make more accessible to Soviet readers some of the new and powerful topological methods originating with J. Leray in 1945, the following papers were translated and collected in one volume: (I) J.-P. Serre, "Homologie singulière des espaces fibrés. Applications", Ann. of Math. (2) **54** (1951), 425–505 [MR **13**, 574]. (II) R. Bott, "On manifolds all of whose geodesics are closed", ibid. **60** (1954), 375–382 [MR **17**, 521]. (III) J.-P. Serre, "Groupes d'homotopie et classes de groupes abéliens", ibid. **58** (1953), 258–294 [MR **15**, 548]. (IV) A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts", ibid. **57** (1953), 115–207 [MR **14**, 490]. (V) A. Borel and J.-P. Serre, "Groupes de Lie et puissances réduites de Steenrod", Amer. J. Math. **75** (1953), 409–448 [MR **15**, 338]. (VI) A. Borel, "La cohomologie mod 2 de certains espaces homogènes" (Chap. 1), Comment. Math. Helv. **27** (1953), 165–197 [MR **15**, 244]. (VII) R. Thom, "Quelques propriétés globales des variétés différentiables", ibid. **28** (1954), 17–86 [MR **15**, 890]. (VIII) H. Cartan, "Variétés analytiques complexes et cohomologie", Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 41–55, Georges Thone, Liège, 1953 [MR **16**, 235]. (IX) J.-P. Serre, "Quelques problèmes globaux relatifs aux variétés de Stein", op. cit., pp. 57–68 [MR **16**, 235]. (X) J.-P. Serre, "Faisceaux algébriques cohérents", Ann. of Math. (2) **61** (1955), 197–278 [MR **16**, 953].

Explanatory footnotes have been generously interspersed by the editors, and almost all papers are followed by lengthy commentaries which amplify statements of the authors or offer alternative proofs.

A. J. Lohwater (Providence, R.I.)

12523:

Shih, Weishu. Sur les systèmes de Postnikov d'un fibré principal. Séminaire C. Ehresmann, 1957/58, exp. no. 5, 8 pp. Faculté des Sciences de Paris, 1959.

This exposé gives a detailed description of the Postnikov resolution for a semisimplical principal fibre bundle with group G and base B. Let $G^{(m)}$ be the mth term in the Moore-Postnikov resolution of G. The mth term in the resolution of the bundle is itself a principal bundle with fibre $G^{(m)}$ and base $B^{\langle m \rangle}$, where $B^{\langle m \rangle}$ is a fibre space with base $B^{(m)}$ and fibre of type $K(\pi, m+1)$, where π is the image of $\pi_{m+1}(B)$ in $\pi_m(G)$.

V. Gugenheim (Baltimore, Md.)

12524:

de Carvalho, Carlos A. A. Sur les deuxièmes obstructions. Séminaire C. Ehresmann, 1957/58, exp. no. 13, 13 pp. Faculté des Sciences de Paris, 1959.

This is an expository account of some of the main results of the thesis of S. D. Liao on the second obstruction to a cross-section of a fibre bundle [see Ann. of Math. (2) **60** (1954), 146–191; MR **15**, 979]. Unfortunately, this account was written before the appearance of a research note by Robert Hermann [see Bull. Amer. Math. Soc. **65** (1959), 5–8; MR **22** #1903] which gives a much neater method of deriving Liao's results.

W. S. Massey (New Haven, Conn.)

12525:

Adams, J. F. On Chern characters and the structure of the unitary group. Proc. Cambridge Philos. Soc. **57** (1961), 189–199.

Let ξ be a stable complex vector bundle over a CW-complex X. Let $m(r)=\prod_p p^{[r/(p-1)]}$, the product being over all primes p. Let $\mathrm{ch}_q(\xi)\in H^{2q}(X;Q)$ be the qth component of the Chern character of ξ. One of the aims of this paper is to prove that if ξ is trivial on the $(2q-1)$-skeleton of X, then $m(r)\mathrm{ch}_{q+r}(\xi)$ is an integral cohomology class. This result follows immediately from property (3) of the following more technical theorem: It is possible to define characteristic classes $\mathrm{ch}_{q,r}(\xi)$ having the following five properties: (1) $\mathrm{ch}_{q,r}(\xi)$ is defined when X is $(2q-1)$-connected; (2) $\mathrm{ch}_{q,r}(\xi)\in H^{2q+2r}(X;Z)$; (3) $k_*(\mathrm{ch}_{q,r}(\xi))=m(r)\mathrm{ch}_{q+r}(\xi)$; (4) if $f\colon X\to X'$, X and X' are $(2q-1)$-connected and $(2q'-1)$-connected respectively, $q\geqq q'$, $q+r=q'+r'$, and ξ' is a stable complex vector bundle over X', then

$$f^*(\mathrm{ch}_{q',r'}(\xi'))=\frac{m(r')}{m(r)}\,\mathrm{ch}_{q,r}(f^*(\xi'));$$

(5) if $r=s(p-1)+t$ with s and t integers such that $s\geqq 0$ and $0\leqq t<p-1$, then

$$\rho_*(\mathrm{ch}_{q,r}(\xi))=\frac{m(r)}{p^s m(t)}\,\chi(\mathscr{P}^s)\rho_*(\mathrm{ch}_{q,t}(\xi)).$$

Here $k\colon Z\to Q$ and $\rho\colon Z\to Z_p$ are the canonical maps, $\mathscr{P}^s$ is the Steenrod reduced pth power, and χ is the canonical anti-automorphism of the Steenrod algebra.

In order to prove this theorem, the author studies parts of the Postnikov system of BU, the classifying space of the infinite unitary group U. In particular, he makes a detailed study of the cohomology of the space obtained by killing the homotopy groups of BU through dimension $2q-1$. The author ends by showing how the k-invariants of BU can be recovered from his characteristic classes.

F. P. Peterson (Oxford)

12526:

Stallings, John. On the loop theorem. Ann. of Math. (2) **72** (1960), 12–19.

The following generalization of the loop theorem of Papakyriakopoulos [same Ann. (2) **66** (1957), 1–26; MR **19**, 761] is given in the sense that the restriction of orientability is removed. Let M be a 3-manifold and let B be a component of its boundary. Let N be a normal subgroup of $\pi_1(B)$; suppose there are elements in Kernel$[\pi_1(B)\to\pi_1(M)]$ which are not in N. Then there is a simple loop $C\subset B$, such that C bounds a nonsingular disk in M, and such that $C(B)\notin N$. The proof is done by using double coverings. A counterexample is further given to the following conjecture related to the above [cf. H. Kneser, Jber. Deutsch. Math. Verein. **38** (1929), 248–260] by using the lens space $L(6,1)$: If M is a 3-manifold, and T is a semilinear 2-manifold in M, and if $\pi_1(T)\to\pi_1(M)$ has nontrivial kernel, then there exists a nonsingular disk Δ in M, such that $\Delta\cap T=C=\mathrm{Bd}\Delta$, with $C(T)\neq 1$.

H. Terasaka (Osaka)

12527:

Weier, Joseph. Ueber Abbildung vierdimensionaler in zweidimensionale Mannigfaltigkeiten. Math. Japon. **5** (1958/59), 83–92.

Let P be a 4-dimensional, Q a 2-dimensional, orientable closed euclidean manifold. Let f be a continuous mapping from P into Q, and denote by F its homotopy class. Arbitrarily close to f there is a mapping f' in F such that the set of points p in P where $f(p)=f'(p)$ is a 2-dimensional finite euclidean polygon A. Let K be a simplicial decomposition of A, denote the oriented 2-simplexes of K by x_t, and let ξ_t be the degree of x_t with respect to (f,f'); then $\sum\xi_t x_t$ is an integral cycle. This paper sets forth to prove that if such an f' exists arbitrarily close to f for which $\sum\xi_t x_t$ bounds in P, then there is an f^* in F arbitrarily close to f and such that $f^*(p)\neq f(p)$ for every point p in P.

P. V. Reichelderfer (Columbus, Ohio)

12528:

Raymond, F. A. A note on the local "C" groups of Griffiths. Michigan Math. J. **7** (1960), 1–5.

The author proves the following theorem. If the locally compact Hausdorff space X is r-lc and $(r+1)$-lc at $x\in X$, then the local group $C_r(x)$ exists at x, is stable, and is naturally isomorphic to $H_{r+1}(X, X-x)$. Here H denotes any augmented exact homology theory defined on the category of locally compact Hausdorff pairs, and the concepts $C_r(x)$ and "lc" are relative to H. The group $C_r(x)$ is that defined by the reviewer [Trans. Amer. Math. Soc. **89** (1958), 201–244; MR **21** #872], where the above theorem is proved in an abstract, and is a Vietoris form [op. cit., § 1.6 and p. 238, (iv)]. Some examples are discussed by the author, and he shows how the theorem can be used to define generalised manifolds without the C_r groups.

H. B. Griffiths (Birmingham)

12529:

Palais, Richard S. A covering homotopy theorem and the classification of G-spaces. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 857–859.

A G-space is a locally compact, second countable, Hausdorff space with a fixed action of G on X. The author describes a classification for G-spaces with a finite number of orbit types. Equivalence classes of G-spaces with fixed orbit space Z correspond to Σ-homotopy classes of Σ-maps of Z into a universal orbit space, where the prefix "Σ-" is an appropriate restriction for keeping track of the various orbit types. The universal orbit space is defined in terms of the topological join. The use of "reduced join" and "$*$" in this context is unfortunate, for it is not that which is common in homotopy theory.

J. Stasheff (Cambridge, Mass.)

DIFFERENTIAL TOPOLOGY

See also 12131, 12132, 12244.

12530:

Thomas, Emery. On the cohomology of the real Grassmann complexes and the characteristic classes of n-plane bundles. Trans. Amer. Math. Soc. **96** (1960), 67–89.

The Pontrjagin squares of the Stiefel-Whitney classes have been determined by W. T. Wu [see Amer. Math. Soc. Transl. (2) **11** (1959), 155–172; MR **22** #5981]. Using this computation, Wu proved that the Pontrjagin classes reduced mod 4 of a differentiable manifold are invariants of the homotopy type of the manifold. In the proof of his formulas, Wu used explicit cellular decompositions of the real Grassmann manifolds, and as a result, his proofs are extremely complicated and difficult to follow.

In the present paper the author gives a new proof of Wu's above-mentioned results, without the use of any cellular decompositions of the Grassmann varieties. He uses only standard, well known facts regarding the cohomology of the classifying spaces for the orthogonal groups and unitary groups together with the fundamental properties of the Pontrjagin squares in his proof.

W. S. Massey (New Haven, Conn.)

12531:

Milnor, John W.; Kervaire, Michel A. Bernoulli numbers, homotopy groups, and a theorem of Rohlin. Proc. Internat. Congress Math. 1958, pp. 454–458. Cambridge Univ. Press, New York, 1960.

Eine zusammenhängende differenzierbare Mannigfaltigkeit M^k mit Basispunkt x_0 heißt fast-parallelisierbar, wenn $M^k - x_0$ parallelisierbar ist. Rohlin [Dokl. Akad. Nauk SSSR **84** (1952), 221–224; MR **14**, 573] hat gezeigt, daß die Pontrjaginsche Zahl $p_1[M^4]$ einer fast-parallelisierbaren M^4 durch 48 teilbar ist. (Für eine kompakte orientierte differenzierbare M^4 ist das Verschwinden der zweidimensionalen Stiefel-Whitneyschen Klasse mit Fast-Parallelisierbarkeit gleichbedeutend.) Die Verff. studieren, welche Pontrjaginschen Zahlen $p_n[M^{4n}]$ für eine fast-parallelisierbare M^{4n} auftreten können. Dazu wird der Homomorphismus $J\colon \pi_{4n-1}(\mathrm{SO}(m)) \to \pi_{4n+m-1}(S^m)$ im stabilen Bereich $m > 4n$ benutzt und eine Beziehung zwischen dem Hindernis, die Parallelisierung auf ganz M^{4n} auszudehnen, und der Pontrjaginschen Zahl bewiesen. (Nach Bott [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 933–935; MR **21** #1588] ist $\pi_{4n-1}(\mathrm{SO}(m))$, $m > 4n$, unendlich-zyklisch.) Ergebnis: $p_n[M^{4n}]$ ist für jede fast-parallelisierbare M^{4n} durch $j_n a_n(2n-1)!$ teilbar, wo j_n die Ordnung von $J\pi_{4n-1}(\mathrm{SO}(m))$, $m > 4n$, ist und $a_n = 2$ für ungerades n und $a_n = 1$ für gerades n. Es existiert eine fast-parallelisierbare M_0^{4n} mit $p_n[M_0^{4n}] = j_n a_n(2n-1)!$.

Dieses Ergebnis kann herangezogen werden, um Auskunft über j_n zu erhalten. In den Arbeiten von Borel-Hirzebruch [Amer. J. Math. **82** (1960), 491–504; MR **22** #11413] und Atiyah-Hirzebruch [Bull. Amer. Math. Soc. **65** (1959), 276–281; MR **22** #989] wurde nämlich bewiesen, daß das $\hat{A}$-Geschlecht einer M^{4n}, welches eng mit dem Toddschen Geschlecht zusammenhängt, eine ganze Zahl ist, falls die zweite Stiefel-Whitneysche Klasse von M^{4n} verschwindet, und unter der gleichen Voraussetzung eine gerade ganze Zahl ist, wenn n ungerade ist. Nun ist für eine fast-parallelisierbare M^{4n}

$$\hat{A}(M^{4n}) = -(B_n/2(2n)!)\cdot p_n[M^{4n}],$$

wo B_n die n-te Bernoullische Zahl ist. Es folgt jetzt, daß die Ordnung j_n der stabilen Gruppe $J\pi_{4n-1}(\mathrm{SO}(m))$ durch den Nenner der rationalen Zahl $B_n/4n$ teilbar ist. Dieser Nenner ist für $n = 1, 2, 3$ gleich 24 bzw. 240 bzw. 504. Aus Sätzen über Homotopiegruppen von Sphären folgt, daß j_n in diesen Fällen gleich den angegebenen Zahlen ist und daß J surjektiv ist. Es wird vermutet, daß j_n für alle n gleich dem Nenner von $B_4/4n$ ist. Die Sätze von Staudt ermöglichen es, diesen Nenner zu berechnen.

F. Hirzebruch (Bonn)

12532:

Suzuki, Haruo. Multiplications in Postnikov systems and their applications. Tôhoku Math. J. (2) **12** (1960), 389–399.

Let $X^{(m)}$ be the mth Postnikov complex of the Thom space $M(\mathrm{O}(n))$. In this paper it is shown that in general $X^{(m)}$ is an H-space for $m < 2n$ and that, when $n = 2$, $X^{(5)}$ is an H-space. These facts are used to obtain theorems concerning the realizability of the sum of two realizable cohomology classes with Z_2 coefficients on a manifold. The proofs of these later theorems are based on the observation that if w is a primitive cohomology class on an H-space A and f, $g\colon X \to A$, then $f^*w + g^*w = (fg)^*w$, where $(fg)(x) = f(x)g(x)$.

E. H. Brown (Waltham, Mass.)

12533:

Tao, Junzo. A note on the Milnor's invariant λ' for a homotopy 3-sphere. Proc. Japan Acad. **36** (1960), 310–312.

Theorem 1 states that $\lambda'(M) = 0$ if M is a homotopy 3-sphere. The proof outlined is incorrect; the connected sum $M \# M$ is confused with $M \# (-M)$, where M and $-M$ have opposite orientations. Theorem 2 (which is correctly proved) states that if every homotopy 3-sphere that is embedded semi-linearly in the 4-sphere S^4 is homeomorphic to S^3, then any homotopy 3-sphere is homeomorphic to S^3.

M. W. Hirsch (Berkeley, Calif.)

12534:

Munkres, James. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. Ann. of Math. (2) **72** (1960), 521–554.

Ce travail constitue l'exposé de résultats annoncés dans une note antérieure [Bull. Amer. Math. Soc. **65** (1959), 332–334; MR **22** #3005]. Soient M une n-variété (à bord) et L un sous-ensemble fermé de M, de dimension $m < n$, muni d'une triangulation (K, h) (K complexe simplicial et h homéomorphisme de la réalisation géométrique $|K|$ sur L) de classe C^2 et induisant une triangulation sur $L \cap \partial M$. Soit N une autre n-variété. On appelle difféomorphisme modulo L une application $f\colon M \to N$ telle que (a) f est un homéomorphisme; (b) sa restriction à $M - L$ est un difféomorphisme; (c) f vérifie en outre des conditions de régularité au voisinage de chaque simplexe de L. On se propose de résoudre le problème fondamental suivant: approcher f par un difféomorphisme modulo L^{m-1}, $(m-1)$-squelette de L. Soient par exemple M et M' deux n-variétés munies de triangulations (K, h), (K', h') de classe C^2; elles sont dites combinatoirement équivalentes s'il existe un isomorphisme F entre ces triangulations. Cet isomorphisme induit entre

simplexes correspondants une transformation linéaire, donc difféomorphe, mais avec des irrégularités sur les faces. L'homéomorphisme $F: M \to N$ est un difféomorphisme modulo K^{n-1} que l'on cherche à approcher par des difféomorphismes modulo les squelettes K^r, $r=n-2, \cdots, 0, -1$. Lorsque cette opération peut être menée à bien, on est assuré que M et N sont difféomorphes. A noter que si la "Hauptvermutung" est vraie, il en résulte que des variétés homéomorphes sont combinatoirement équivalentes.

L'auteur montre que le problème fondamental conduit à un obstacle $\lambda_m f$ qui est une m-chaîne de L à coefficients tordus dans le groupe abélien Γ^{n-m} de Thom-Milnor. Cette chaîne induit un cycle dans $C_m(L, L \cap \partial M; \Gamma^{n-m})$, et si $\lambda f = 0$ on peut approcher f par un difféomorphisme $g: M \to N$ modulo L^{m-1}. En outre, si $\lambda_{m-1} g$ induit la classe nulle dans $H_{m-1}(L, L \cap \partial M; \Gamma^{n-m+1})$, l'approximation g peut-être choisie de manière à vérifier la condition $\lambda_{m-1} g = 0$ et à coïncider avec f en dehors d'un voisinage donné de L: on pourra donc alors approcher f par un difféomorphisme modulo L^{m-2}.

On trouvera dans un premier paragraphe un exposé complet (inexistant jusqu'à présent) sur le groupe Γ^n. Celui-ci s'obtient en considérant les groupes Diff B^n et Diff S^{n-1} des difféomorphismes conservant l'orientation de la boule B^n et de la sphère S^{n-1} munis de la C^1-convergence compacte et de l'homomorphisme évident du premier dans le second. On forme alors les groupes $\pi_0(\text{Diff } B^n)$ et $\pi_0(\text{Diff } S^{n-1})$ des composantes connexes par arcs et on vérifie que le dernier est abélien. Le groupe Γ^n est défini comme conoyau de l'homomorphisme $\pi_0(\text{Diff } B^n) \to \pi_0(\text{Diff } S^{n-1})$. L'homomorphisme Diff $B^n \to$ Diff S^{n-1} admet un conoyau qui n'est autre que Γ^n.

Comme applications, notons notamment la démonstration d'une conjecture de E. E. Moise suivant laquelle un homéomorphisme entre variétés différentiables de dimension 3 peut-être approché autant que l'on veut par un diffémorphisme. L'auteur donne aussi des démonstrations de résultats de Thom et Milnor concernant les structures différentiables de R^n, B^n et S^n. Si elles sont combinatoirement équivalentes aux structures usuelles, elles sont uniques dans les deux premiers cas; pour S^n elles correspondent aux éléments de Γ^n [R. Thom, #12535]. *P. Dedecker* (Caracas)

12535:

Thom, R. Les structures différentiables des boules et des sphères. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 27–35. Centre Belge Rech. Math., Louvain, 1959.

The main results of this paper are the following two theorems: (a) Suppose that D^n is a differentiable manifold with boundary which is known to be homeomorphic to the closed unit ball in n-dimensional Euclidean space. Suppose further that D^n admits a differentiable triangulation which is combinatorially equivalent to the usual triangulation of the unit ball. Then D^n is diffeomorphic to the closed unit ball in Euclidean n-space. Corollary: If the "Hauptvermutung" is true for triangulations of D^n, then there is a unique differentiable structure on D^n. (b) Every differentiable manifold homeomorphic to an n-sphere which admits a differentiable triangulation combinatorially equivalent to the usual triangulation of the n-sphere may be obtained by identifying the boundaries of two n-dimensional balls (having the usual differentiable structure) by means of a diffeomorphism.

It is impossible to give a satisfactory idea of the method of proof used by the author in a short review. The author contents himself with a very sketchy proof in the paper itself. {Reviewer's note: These two results have subsequently been obtained by J. Munkres by means of his obstruction theory; see Bull. Amer. Math. Soc. **65** (1959), 332–334; MR **22** #3005; and #12534 above.}

W. S. Massey (New Haven, Conn.)

12536:

Thom, R. Des variétés triangulées aux variétés différentiables. Proc. Internat. Congress Math. 1958, pp. 248–255. Cambridge Univ. Press, New York, 1960.

The problem here is to define a differentiable structure on a simplicial complex K which is a topological manifold, the structure being defined step-by-step on the successive skeletons of K. Suppose that a differentiable structure $\mathscr{S}$ has already been defined on a neighborhood of the k-skeleton and is to be extended to a neighborhood of the $(k+1)$-skeleton. This problem reduces to that of extending over a neighborhood of a simplex σ^{k+1} a differentiable structure $\mathscr{S}$ given on a neighborhood of the boundary $\partial\sigma^{k+1}$. This problem can be thought of in two stages: (1) extension of the tangential structure of $\partial\sigma$ to σ; (2) extension of the normal vector bundle over σ. (2) could be solved if it were known that the space of semi-linear homeomorphisms of a subdivision of a simplex on itself, reducing to the identity on the boundary is aspherical. The obstruction to (1) is an element of the group of diffeomorphisms of S^k modulo those extending to the $(k+1)$-ball. Also a sketch proof of the following is given: Let M be a differentiable manifold homeomorphic to $V \times I$, where V is a differentiable manifold. Let K be a differentiable triangulation of V and let M have a differentiable triangulation isomorphic to $K \times I$. Then $V \times \{0\}$ and $V \times \{1\}$ are diffeomorphic and M is diffeomorphic to $V \times I$. *A. H. Wallace* (Bloomington, Ind.)

12537:

Thom, René. Remarques sur les problèmes comportant des inéquations différentielles globales. Bull. Soc. Math. France **87** (1959), 455–461.

Let M^n, V^p be differentiable manifolds, $J^r(M, V)$ the bundle of r-jets over $M \times V$. One can define on $J^r(M, V)$ a canonical Pfaffian system Ω. For example, on $J^1(R, R)$ with coordinates x, y, p, q, this sytem is defined by the forms $dy - p\,dx$, $dp - q\,dx$. For an open set $O \subset J^r(M, V)$, let $G(O)$ be the singular complex of O, $G_\Omega(O)$ the subcomplex of differentiable singular simplices which are integrable with respect to Ω (i.e., every form in Ω vanishes on these simplices). The main theorem of this paper states that for any open set $O \subset J^r(M^n, V^p)$ the homomorphism on the homology groups induced by the inclusion $G_\Omega(O) \to G(O)$ is bijective in dimensions less than n and surjective in dimension n. Moreover, on any manifold, for a system Ω locally equivalent to a system of the above type, a similar result is valid.

{The author has said to the reviewer that, although he believes his proof to be valid for $r=1$, there seem to be further difficulties in case $r>1$.}

S. Smale (Berkeley, Calif.)

12538:

Reeb, G. Trois problèmes de la théorie des systèmes dynamiques. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 89–94. Centre Belge Rech. Math., Louvain, 1959.

The author lists three typical problems in "dynamical systems". I. The author's viewpoint is as follows: In the classical theory of general dynamical systems (à la Birkhoff) one is given a group acting on a topological space. The trajectories (orbits) can be regarded as equivalence classes, two points being equivalent if they differ by the action of the group. The author defines a "general dynamical system" as a topological space with an equivalence relation defined on each open set and subject to suitable axioms. (The definition given here is more general than the one given in his paper in Ann. Inst. Fourier. Grenoble **6** (1955/56), 89–112 [MR **18**, 407].) The problem is then to transcribe some of the results of dynamical systems (in particular a theorem of Hilmy) into this setting. II. What manifolds V_{2n+1} can carry a contact structure all of whose orbits are closed? III. On a manifold with boundary whose Euler characteristic does not vanish, can there exist a non-zero vector field nowhere tangent to the boundary such that an orbit both enters and leaves V_n? If so, the set of orbits staying inside is extremely complicated. The answer, according to a forthcoming paper by the author, is affirmative.

S. Sternberg (Cambridge, Mass.)

12539:

Gherardelli, Francesco. Sui gruppi dell'equivalenza e della torsione algebrica sulle varietà kähleriane e algebriche. Ann. Mat. Pura Appl. (4) **46** (1958), 71–95.

Let X be a compact connected complex manifold with fundamental group π_1 and universal covering space U. Let K be the commutator subgroup of π_1, so that $G=\pi_1/K\cong H_1(X, Z)$. Then $V=U/K$ is the maximal abelian covering of X. Let $F(V)$ be the field of meromorphic functions on V, $A(V)$ the subgroup of the multiplicative group of $F(V)$ consisting of all functions f such that $|\sigma f|=|f|$ for all $\sigma\in G$. Put $\chi_f(\sigma)=\sigma f/f$; then χ_f is a homomorphism of G into the group of complex numbers of unit modulus. Now define subgroups A_0, A_1 by

$$A_0=\{f\in A(V)|\ \chi_f=1\},$$

$$A_1=\{f\in A(V)|\ \chi_f=1 \text{ for all } \sigma\in G \text{ of finite order}\}.$$

The author defines $\mathcal{G}_a=A_1/A_0$ to be the group of algebraic equivalence of X and $\mathcal{T}=A/A_1$ to be the group of algebraic torsion of X. If one replaces $A(V)$ by the subgroup

$$A^{(r)}=\{f\in A(V)|\ \chi_f^n=1 \text{ for some } n \text{ depending on } f\},$$

one obtains in a similar way the group $\mathcal{G}_a^{(r)}$ of rational algebraic equivalence.

The author studies these groups on algebraic manifolds and, more generally, on Kähler manifolds whose Albanese varieties are algebraic. He also gives an alternative ideal-theoretic definition of the group $\mathcal{G}_a^{(r)}\times\mathcal{T}$, and this definition applies to algebraic varieties over arbitrary fields.

M. F. Atiyah (Oxford)

12540:

Rothaus, Oscar S. Domains of positivity. Abh. Math. Sem. Univ. Hamburg **24** (1960), 189–235.

A domain of positivity (self-dual cone) is a non-empty open set D in R^n for which there exists a non-degenerate bilinear form B such that $x\in D$ if and only if $B(x, y)>0$ for all $y\in\bar{D}$, $y\neq 0$. Domains of this type have been studied by M. Koecher [Amer. J. Math. **79** (1957), 575–596; MR **19**, 867]. The linear transformations W of R^n that map D onto itself form a Lie group G, called the group of automorphisms of D. $W\to W^*=(W^t)^{-1}$ is an involutive automorphism of G (W^t denotes the transpose with respect to B). If G is transitive, D is said to be homogeneous. Koecher showed the existence of a function N on D such that $(1/N(x))dx_1\wedge\cdots\wedge dx_n$ is a volume on D invariant under G. In case of a homogeneous D he also defined a G-invariant metric on D and an involutive isometry $x\to x^*$ of D such that $N(x)N(x^*)=1$ and $(Wx)^*=W^*x^*$ for all $W\in G$. In the present paper the author studies the tube T_D over D, i.e., the complex domain $D\times iR^n$ in $C^n=R^n\times iR^n$. The first important result is the determination of the Bergman kernel of T_D, which turns out to be

$$\frac{1}{\pi^n}\frac{\Delta(2)}{\Delta(1)}\frac{1}{N(z+\bar{w})^2}$$

(Δ is the "gamma function" of D, introduced by Koecher, N is extended to T_D by analytic continuation). The proof is based on Fourier analysis, and is similar to that of Bochner's result establishing a Cauchy formula for tube domains [Ann. of Math. (2) **45** (1944), 686–707; MR **6**, 123]. It seems to the reviewer that for this result neither the homogeneity nor the self-duality of the cone D are essential. In the following the author studies homogeneous domains of positivity with a positive definite B. In this case he shows that $x\to x^*$ has an isolated point p, so D is a symmetric space. He also shows that $K(x)$, the negative Jacobian of $x\to x^*$, is for all x an element of G, thus answering a question posed by Koecher; actually it turns out that the $K(x)$ are exactly the transvections based at p. Next it is shown that the map $x\to x^*$ can be extended to a holomorphic involution $z\to z^*$ of T_D, which is an isometry for the Bergman metric, and for which p is still an isolated fixed point. So T_D is a hermitian symmetric space, and as it is easy to show, it is holomorphically equivalent to a bounded domain. (It is to be noted that not all bounded symmetric domains can be obtained from this construction, only those whose Bergman-Šilov boundary has real dimension n.) Another basic result of the author is that in case of a positive definite B, $N(z)^2$ is a polynomial. The rest of the paper is concerned with an explicit description of the Lie algebra of G, which is done without making use of the general theory of symmetric spaces.

A. Korányi (Berkeley, Calif.)

12541:

Kuhlmann, Norbert. Projektive Modifikationen komplexer Räume. Math. Ann. **139**, 217–238 (1960).

Die vorliegende Arbeit liefert einen Beitrag zur Theorie der eigentlichen Modifikationen komplexer Räume. Eine eigentliche holomorphe Abbildung $\varphi: X'\to X$ eines komplexen Raumes X' auf einen komplexen Raum X heisst eine eigentliche Modifikation von X, wenn es in X' eine dünne analytische Menge N' gibt, sodass $N:=\varphi(N')$ dünn in X ist und $X'-N'$ vermöge φ biholomorph auf $X-N$ abgebildet wird. Die bekanntesten eigentlichen Modifikationen sind die sog. σ-Modifikationen. Eine σ-Modifikation

"bläst" eine singularitätenfrei in X liegende dünne analytische Menge N zu einer in X' dünnen analytischen Menge N' auf und lässt $X-N$ unverändert. Der Verfasser gibt in der vorliegenden Arbeit allgemeinere Rezepte zur Erzeugung eigentlicher Modifikationen, die es insbesondere ermöglichen, beliebige (nicht notwendig singularitätenfrei in X liegende) dünne analytische Mengen "aufzublasen". Dies leisten z.B. die vom Verfassern eingeführten monoidalen Modifikationen, die das analytische Analogon zu den monoidalen Transformationen der algebraischen Geometrie im Sinne O. Zariskis sind. Es wird bewiesen, dass im Falle algebraischer Varietäten über dem Grundkörper der komplexen Zahlen die monoidalen Transformationen und die monoidalen Modifikationen analytisch äquivalente Prozesse sind. Die σ-Modifikationen sind genau die monoidalen Modifikationen mit "singularitätenfrei eingebetteter Basis".

Das benutzte Konstruktionsverfahren lässt sich noch verallgemeinern. Der Verfasser gelangt da bei zu den von ihm sogenannten "projektiven Modifikationen". Er zeigt, dass jede projektiv-einbettbare eigentliche Modifikation X' eines projektiv-einbettbaren komplexen Raumes X eine projektive Modifikation ist. Umgekehrt gilt: jede projektive Modifikation X' eines projektiv-einbettbaren komplexen Raumes X ist wiederum projektiv-einbettbar. Dieses Resultat zeigt insbesondere, dass sicher nicht alle eigentlichen Modifikationen projektive Modifikationen sind, da es Beispiele nicht projektiv-einbettbarer eigentlicher Modifikationen X' projektiv-einbettbarer komplexer Räume X gibt.

An Hilfsmitteln werden vor allem Sätze über holomorphe und meromorphe Abbildungen sowie Ergebnisse von J. P. Serre über den Zusammenhang zwischen der analytischen und algebraischen Struktur algebraischer Varietäten über den komplexen Zahlen als Grundkörper herangezogen. *R. Remmert* (Erlangen)

12542:

Srinivasacharyulu, Kilambi. Sur la déformation des structures complexes. C. R. Acad. Sci. Paris **251** (1960), 2648–2649.

In this note the author proves that every deformation of a complex torus is again a complex torus. He also proves the following theorem. Let G be a complex abelian Lie group, B a complex manifold, P a differentiable principal fibre bundle over B with group G, and let I_0, I_1 be two complex structures on P compatible with the fibration. Then there exists a differentiable homotopy I_t of complex structures on P. *M. F. Atiyah* (Oxford)

12543:

Porteous, I. R. Blowing up Chern classes. Proc. Cambridge Philos. Soc. **56** (1960), 118–124.

Es sei X eine n-dimensionale projektive algebraische Mannigfaltigkeit und Y eine singularitätenfreie irreduzible Untermannigfaltigkeit der Dimension $m \leqq n-2$. Wir betrachten die monoidale Transformation von X entlang Y und erhalten eine n-dimensionale singularitätenfreie Mannigfaltigkeit X', die eine $(n-1)$-dimensionale Untermannigfaltigkeit Y' enthält, welche über Y in $(n-m-1)$-dimensionale projektive Räume gefasert ist (mit der Projektion $g: Y' \to Y$). Wir haben das kommutative Diagramm

$$\begin{array}{ccc} X' & \xleftarrow{j} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xleftarrow{i} & Y \end{array}$$

wo j und i die Einbettungen sind. $f: X'-Y' \to X-Y$ ist bijektiv. Der Verfasser stellt einem Zusammenhang her zwischen den Chernschen Klassen von X und denen der "aufgeblasenen" Mannigfaltigkeit X'. Im Grothendieckschen Ring $K(X')$ [vgl. A. Borel und J. P. Serre, Bull. Soc. Math. France **86** (1958), 97–136; MR **22** #6817] hat man die Gleichung

$$f^!T_X - T_{X'} = j_!(g^!E - L), \tag{1}$$

wo T_X bzw. $T_{X'}$ das Tangentialbündel von X bzw. X' und E bzw. L das Normalbündel von Y in X bzw. von Y' in X' ist. (1) kann mit Hilfe der lokalen Gleichungen der Dilatation bewiesen werden. Wendet man auf (1) den Chernschen Charakter an, dann ergibt sich

$$f^* \operatorname{ch} X - \operatorname{ch} X' = \operatorname{ch} j_!(g^!E - L),$$

wo mit ch X abkürzend der Chernsche Charakter des Tangentialbündels von X bezeichnet wird, ebenso für X'. Nun wendet der Verfasser den Satz von Riemann-Roch-Grothendieck (vgl. Borel-Serre, loc. cit.) an, um ch und $j_!$ "vertauschen" zu können. Dies ergibt

$$f^* \operatorname{ch} X - \operatorname{ch} X' = j_*[(g^* \operatorname{ch} E - e^v)((1-e^{-v})/v)], \tag{2}$$

wo v die zweidimensionale Chernsche Klasse von L ist. Rechnet man den Chernschen Charakter in die Chernsche Klasse um, dann ergibt sich modulo Torsion folgende Formel:

$$\begin{aligned} c(X') = f^*c(X) + j_*\{(g^*c(Y))(1/v) \\ \times[(1+v)\sum_i (1-v)^i g^* c_{n-m-i}(E) - g^*c(E)]\}. \end{aligned} \tag{3}$$

Der Verfasser beweist (3) auch ohne die Einschränkung "modulo Torsion", allerdings nur wenn die Charakteristik des Grundkörpers gleich 0 ist. Die Formel (3) wurde von Todd und Segre vermutet [J. A. Todd, Proc. London Math. Soc. (2) **47** (1941), 81–100; MR **3**, 14]; van de Ven [Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 571–578; MR **18**, 762] bewies (3) in Spezialfällen, konnte aber gewisse Schwierigkeiten nicht überwinden, was dem Verfassern nun mit Hilfe des Grothendieckschen Satzes gelungen ist. Für das Aufblasen beliebiger kompakter komplexer Mannigfaltigkeiten lassen sich die Schwierigkeiten ebenfalls überwinden, sobald man den Riemann-Rochschen Satz im Sinne von Grothendieck zur Verfügung hat. *F. Hirzebruch* (Bonn)

12544:

Grauert, Hans. Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. Inst. Hautes Études Sci. Publ. Math. No. 5 (1960), 64 pp.

Cet ouvrage est consacré à la démonstration du résultat suivant: (Théorème fondamental I) Soient X et Y des espaces analytiques complexes, π un morphisme propre de X dans Y, F un faisceau analytique cohérent sur X. Alors les faisceaux images $R^q\pi_*F$ sont cohérents.

La technicité de cette démonstration en rend la lecture ardue, voire fastidieuse quand on est amené à considérer

un grand nombre de recouvrements strictement plus fins les uns que les autres. Mais ce théorème doit être considéré comme l'un des résultats fondamentaux de la théorie des faisceaux analytiques cohérents.

Dans les deux premiers paragraphes, l'auteur rappelle les fondements de la géométrie analytique complexe et de la théorie des faisceaux analytiques cohérents. Il admet des nilpotents dans les anneaux locaux de ses espaces analytiques, ce qui dans la suite simplifie certains énoncés mais n'intervient pas essentiellement dans la démonstration.

Il continue dans le § 3 en introduisant une pseudonorme sur l'espace des sections du faisceau F sur certains ouverts U de l'espace X. Si par exemple $U = K \times G$, où K est un polycylindre de $\mathbf{C}^m$ de polyrayon $\rho = (\rho_1, \cdots, \rho_m)$ et G un ouvert de $\mathbf{C}^n$, et si $F = \mathcal{O}$ est le faisceau des anneaux locaux, une section $f \in \Gamma(U; F)$, i.e., une fonction holomorphe sur $K \times G$, se développe en série :

$$f(t, x) = \sum \left(\frac{t_1}{\rho_1}\right)^{i_1} \cdots \left(\frac{t_m}{\rho_m}\right)^{i_m} f_{i_1, \cdots, i_m}.$$

On pose alors

$$\|f_i\|_G = \sup_{x \in G} |f_i(x)|, \quad \|f\|_{K \times G} = \sup_{i_1, \cdots, i_m} \|f_{i_1, \cdots, i_m}\|_G.$$

Dans tous les cas, l'espace des sections de pseudonorme finie est un espace de Banach. Le fait d'avoir des espaces de Banach, et pas seulement de Fréchet, sera essentiel au point crucial de la démonstration (§ 5, no. 5, p. 53). Vient ensuite une série de théorèmes précisant le comportement, vis-à-vis de cette norme, de résultats classiques tels que le théorème des voisinages privilégiés (Satz 1) ou des conséquences du théorème B de H. Cartan.

Avec le § 4, on s'enfonce dans la technique des raffinements de recouvrements. On en ressort avec un théorème (Satz 8) du type de celui de Leray sur les recouvrements acycliques.

Les §§ 5 et 6 contiennent la démonstration proprement dite du théorème fondamental I. De cette démonstration découle également un autre résultat, analogue à un résultat de Grothendieck en géométrie algébrique : (Théorème fondamental II) Dans les hypothèses du théorème fondamental I, soit $\mathfrak{m}$ l'idéal maximal d'un point $y \in Y$. On a alors, quand $r \to \infty$,

$$\varprojlim ((R^q \pi_* F)_y \otimes \mathcal{O}/\mathfrak{m}^r) = \varprojlim H^q(F \otimes \pi^* \mathcal{O}/\mathfrak{m}^r).$$

Le § 7 donne quelques applications. *A. Douady* (Paris)

12545 :

Novikov, S. P. Some problems in the topology of manifolds connected with the theory of Thom spaces. Dokl. Akad. Nauk SSSR **132** (1960), 1031–1034 (Russian); translated as Soviet Math. Dokl. **1**, 717–720.

In this brief note the author announces results on the cobordism theories corresponding to the "stable Thom complexes" MSO, MU, MSp. The ideas and methods are comparable with, but apparently largely independent of, the work of J. Milnor [Abstract 547–26, Amer. Math. Soc. Not. **5** (1958), 457; Amer. J. Math. **82** (1960), 505–521; MR **22** #9975].

The note begins with the geometrical aspect of cobordism theories. Let G be a subgroup of $O(n)$. A smooth manifold W^i embedded in R^{n+i} is said to be "G-equipped" if its normal bundle is given a G-bundle structure. By defining a suitable equivalence relation between such G-equipped manifolds, one obtains cobordism groups, written $V_n^i(G)$. Following Thom [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890] one finds that $V_n^i(G) \simeq \pi_{n+i}(M_n(G))$, where $M_n(G)$ denotes a Thom complex. The author confines his attention to the groups $V_n^i(\mathrm{SO}(n))$, $V_{2n}^i(\mathrm{U}(n))$, $V_{4n}^i(\mathrm{Sp}(n))$ for n sufficiently large; these groups are independent of n, and he calls them V_{SO}^i, V_{U}^i and V_{Sp}^i. They correspond to the stable homotopy groups of the stable Thom complexes MSO, MU and MSp. The theoretical work includes a discussion of the products in the graded cobordism rings $V_G = \sum_i V_G^i$ ($G = \mathrm{SO}, \mathrm{U}, \mathrm{Sp}$); these products may be induced by product maps $\mu : MG \times MG \to MG$ ($G = \mathrm{SO}, \mathrm{U}, \mathrm{Sp}$).

The geometrical approach sketched above employs normal bundles, following Thom (reference above); it is simple, elegant and natural. It may be worth remarking, however, that for some purposes it is necessary to pass from the normal bundle of W^i to its tangent bundle. (For example, see Hirzebruch, Proc. Internat. Congress Math. 1958, pp. 119–136, especially pp. 122–127 [Cambridge Univ. Press, New York, 1960]. Here one has $G = \mathrm{U}$.) Unfortunately, in Milnor's paper (cited above) all consideration of the geometrical aspects is postponed to Part II, which has not yet appeared. Such ideas also figure in Milnor's lecture to the Zürich Colloquium on Differential Geometry and Topology, 1960; but this also has not yet appeared in print.

The author's results on the rings V_G are as follows.

Theorem 1 : The rings V_{SO}, V_{U} and V_{Sp} do not have p-torsion for $p > 2$, and the ring V_{U} has no 2-torsion. The quotient ring $V_{\mathrm{SO}}/(2\text{-torsion})$ is a polynomial ring on generators of dimension $4i$. The ring V_{U} is a polynomial ring on generators of dimension $2i$. The algebras $V_{\mathrm{Sp}} \otimes Z_p$ ($p > 2$), $V_{\mathrm{Sp}} \otimes Q$ (where Q denotes the rationals) are polynomial algebras on generators of dimension $4i$.

Theorem 2 : The quotient ring $V_{\mathrm{Sp}}/(2\text{-torsion})$ is not a polynomial ring. Indeed, this quotient ring has generators x, y of dimension 4, 8 such that $x^2 = 4y$.

Following Thom, these results are applied to give a sufficient condition that a homology class in a manifold should be representable by a submanifold; similarly for a homology class in a complex.

The author's methods are as follows. In order to compute the required stable homotopy groups, the first step is to compute the cohomology groups $H^*(MG; Z_p)$, qua modules over the Steenrod algebra A, for $G = \mathrm{SO}, \mathrm{U}, \mathrm{Sp}$. The result is stated as Lemma 1, and occupies sixteen lines. The deepest results are the following. (1) The module $H^*(M\mathrm{Sp}; Z_2)$ is the direct sum of a stated number of modules A/J, where J is the two-sided ideal in A generated by Sq^1 and Sq^2. (2) The module $H^*(M\mathrm{SO}; Z_2)$ is the direct sum of submodules each isomorphic to A or to $A/(A\mathrm{Sq}^1)$.

There are apparently two possible methods of proving the latter result. (a) The structure of $H^*(M\mathrm{SO}; Z_2)$ is explicitly known, and one knows how to calculate the Steenrod squares in it. From this it is presumably possible to deduce result (2) by purely algebraic means, without further appeal to topology. However, the task of constructing such a proof is not easy. If it were done, one would have an independent method of determining V_{SO}. (For example, it follows from (2) that all the 2-torsion of V_{SO} is of order 2.) (b) Conversely, given sufficient information about V_{SO}, one can recover result (2), as was

done by C. T. C. Wall [Bull. Amer. Math. Soc. **65** (1959), 329–331; Ann. of Math. (2) **72** (1960), 292–311; MR **21** #6586; **22** #11403].

From the present note one cannot determine whether the author follows method (a) or (b), although method (a) appears to be much more in character. From a manuscript, however, it appears that the author follows method (b), basing his work on that of Rohlin [Dokl. Akad. Nauk SSSR **89** (1953), 789–792; MR **15**, 53]. It is presumably for this reason that the author does not use result (2) to deduce results on the 2-torsion of V_{SO}.

In all cases except the case $G=SO$, $p=2$, the second step is to compute $\mathrm{Ext}_A(L, Z_p)$ (together with its products) for the various A-modules L which arise. This is done for all cases except the case $G=Sp$, $p=2$, which is left incomplete.

The third step is to apply the reviewer's spectral sequence [Comment Math. Helv. **32** (1958), 180–214; MR **20** #2711]. This leads to the results announced.

When the reviewer first saw these ideas and methods, in Milnor's work, he formed a very high opinion of them; and one cannot grudge the same praise to anyone who comes to them independently. If one wishes to examine the question of priority, one finds that Milnor's first announcement in 1958 (reference above) dealt only with the case $G=SO$, but contained an indication of the method. It was also in 1958 that Milnor's work was presented to the International Congress of Mathematicians. (See the reference to Hirzebruch given above; this presentation concentrated on the case $G=U$.) The work aroused much interest, and (for example) the reviewer was able to borrow, study and copy one of Milnor's manuscripts. The paper of Novikov under review is contained in the issue of the Doklady dated June, 1960. Presumably it should have credit for what it says about the case $G=Sp$ and about the geometrical aspects.

Milnor's full-scale paper (see reference above) is contained in the issue of the American Journal of Mathematics dated July, 1960. In addition to what has been noted above, it contains one-line remarks on the cases $G=Sp$, $p>2$ and $G=Spin$, $p>2$. The case $G=SU$ is proposed as a problem; one hears that Novikov is working on this.

{Editor's remark. The following errata and corrigenda in the translation have been called to the attention of the Editor: page 717, line 10, read "cobordism groups" for "inner homology groups"; page 718, line 4, "stable cohomology groups" for "fixed cohomology groups"; page 718, line 20, "modules" for "a module"; page 718, line 17 "in addition to" for "instead of".}

J. F. Adams (Cambridge, England)

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